# 9. *Boundedness of the Spectrum of a Distribution Function.*

By Tatsuo KAWATA.

Mathematical Institute, Tohoku Imperial University, Sendai.

(Comm. by M. FUJIWARA, M.I.A., Feb. 13, 1939.)

It is generally recognized that the smoothness of a distribution function $\sigma(x)$ depends on the order of vanishing at infinity of its characteristic function (Fourier-Stieltjes transform of $\sigma(x)$)

$$\Lambda(t;\sigma)=\int_{-\infty}^{\infty} e^{itx}\,d\sigma(x)\,. \tag{1}$$

For example, if

$$\Lambda(t;\sigma)=O\left(e^{-c|t|}\right),\qquad c \text{ constant}, \tag{2}$$

then $\sigma(x)$ is analytic. Especially it results that $\sigma(x)$ can not be constant in any interval or in other words the spectrum[1] of $\sigma(x)$ becomes the whole real axis. Connecting with the spectrum the following more general fact will be obtained.

*Let $p(t)$ be defined in $(0, \infty)$ and positive and let $tp'(t)\uparrow\infty$. If*

$$\Lambda(t;\sigma)=O\left(e^{-p(|t|)}\right) \tag{3}$$

*and*

$$\int_1^{\infty}\frac{p(t)}{t^2}\,dt=\infty\,, \tag{4}$$

*then the spectrum of $\sigma(x)$ is the whole line.*

This is known in the case where $\sigma(x)$ is absolutely continuous.[2] In fact, $\sigma'(x)$ becomes indefinitely differentiable and quasi-analytic in the sense of Carlemann-Denjoy. The proof of the above theorem will be done also quite similarly.

Now our purpose is to prove the following fact concerning the converse problem.

*Let $p(t)$ be positive and increasing. If*

$$\int_1^{\infty}\frac{p(t)}{t^2}\,dt<\infty\,, \tag{5}$$

*then there exists a distribution function $\sigma(x)$ with bounded spectrum and such that*

---

1) The point $x$ is said after A. Wintner the spectrum of a distribution function $\sigma(x)$ if there exist two points $x'$ and $x''$ in any vicinity of $x$ such that $\sigma(x')\neq\sigma(x'')$.

2) In the case where $\sigma(x)$ is absolutely continuous and $\sigma'(x)$ is periodic and the Fourier coefficients satisfy (3), this was proved by Mandelbrojt. Mandelbrojt, Série de Fourier et classes quasi-analytiques, Borel collections, 1936. And in the case of usual Fourier transforms, see:

Ingham, Notes on Fourier transforms, Journal of London Math. Soc., **9** (1934).

Izumi-Kawata, Tohoku Math. Journal, **42** (1937).

S. Takenaka, ibid. **44** (1938).