PAPERS COMMUNICATED

30. *On the Compactness of a Class of Functions.*

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1. In this note we will consider the class of functions defined in the finite interval (a, b).

Let $\mathfrak{F}_c$ be a class of continuous functions defined in (a, b). If any sequence of functions in $\mathfrak{F}_c$ contains a uniformly convergent subsequence, then $\mathfrak{F}_c$ is called compact or compact in (C), where (C) denotes the class of all continuous functions. Arzèla's theorem concerning the compactness of $\mathfrak{F}_c$, is well known, which runs as follows:

Theorem A. *In order that the class $\mathfrak{F}_c$ be compact, it is necessary and sufficient that*

1°. $\mathfrak{F}_c$ is bounded, that is, there is a constant K such that $|f(x)| \leqq K$ for all f in $\mathfrak{F}_c$.

2°. $\mathfrak{F}_c$ is equally continuous, that is, for any positive number δ, there is an $\eta > 0$ such that the oscillation of functions in any interval with length less than η is less than δ.

Instead of (C) we take the class (L^p) $(p \geqq 1)$. Let $\mathfrak{F}_l$ be a class of functions in (L^p). If any sequence in $\mathfrak{F}_l$ contains a mean convergent subsequence with index p, then $\mathfrak{F}_l$ is called compact or compact in (L^p). Fréchet has proved the following theorem[1]:

Theorem B. *In order that the class $\mathfrak{F}_l$ be compact, it is necessary and sufficient that 1°. $\mathfrak{F}_l$ is almost equally continuous and 2°. $\mathfrak{F}_l$ is equally integrable.*

Finally let (S) be the class of all finite measurable functions. Let $\mathfrak{F}_s$ be a class of functions in (S). If any sequence in $\mathfrak{F}_s$ contains a subsequence convergent in measure, then $\mathfrak{F}_s$ is called compact or compact in (S). Fréchet has also proved that[2]

Theorem C. *In order that the class $\mathfrak{F}_s$ be compact, it is necessary and sufficient that 1°. $\mathfrak{F}_s$ is almost equally bounded and 2°. $\mathfrak{F}_s$ is almost equally continuous.*

On the other hand Kolmogoroff[3] has proved the following theorem:

Theorem D. *In order that $\mathfrak{F}_l$ in (L^p) be compact, it is necessary and sufficient that*

1°. $\mathfrak{F}_l$ is bounded, that is, there is a constant K such that

$$\int_a^b |f(x)|^p dx \leqq K$$

for all f in $\mathfrak{F}_l$.

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1) M. Fréchet, Acta de Szeged, **8** (1937).

2) Fréchet, Fund. Math., **9** (1911).

3) A. Kolmogoroff, Göttinger Nachr., 1931. For the detailed literature, see T. Takahashi, Studia Math. **5** (1935).