PAPERS COMMUNICATED

58. *On the Division of a Probability Law.*

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1. If a random variable X is represented as a sum X_1+X_2 of independent variables X_1 and X_2, in other words, if the characteristic function of X

$$f(t)=\int_{-\infty}^{\infty} e^{itx} d\sigma(x)$$

$\sigma(x)$ being the distribution function of X, is represented as a product $f_1(t)f_2(t)$ of characteristic functions $f_1(t)$ and $f_2(t)$ of X_1 and X_2 respectively, X is said to be divisible by X_1 or X_2. The division of X by some random variable is not necessarily determined uniquely what was proved by Gnedenko and Khintchine.[1] That is, there exist characteristic functions $f(t)$, $f_1(t)$, $f_2(t)$ and $f_3(t)$ such that

$$f(t)=f_1(t)f_2(t)=f_1(t)f_3(t)\,, \tag{1}$$

where $f_2(t)$ is not identically equal to $f_3(t)$. But it was shown by P. Lévy that if X is indefinitely divisible, then the division is uniquely determined. The purpose of this paper is to discuss the unicity of divisibility in terms of a distribution function of X.

2. If there exists a t_0 such that $f_2(t_0)\neq f_3(t_0)$, then since a characteristic function is continuous there exists an interval $a<t<b$ in which $f_2(t)\neq f_3(t)$. Since if (1) holds then $f_1(t)\{f_2(t)-f_3(t)\}=0$, in this case $f_1(t)$ or $f(t)$ vanishes in $a<t<b$. Thus the sufficient conditions for the non-vanishing of $f(t)$ in any interval is also the sufficient conditions for the unique determination of division of X. Hence known results[2] on non-vanishing of function yield the following theorems.

Theorem 1. Let $\sigma(x)$ be the distribution function of a random variable X and let $\theta(u)$ be a positive, non-decreasing function defined in $(0, \infty)$ such that

$$\int_1^{\infty} \frac{\theta(u)}{u^2}\, du=\infty\,. \tag{2}$$

If for some constant a (>0)

$$\sigma(-u+a)-\sigma(-u-a)=O\big(\exp(-\theta(u))\big) \tag{3}$$

and X is divisible by some variable, then the quotient is unique.

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1) B. Gnedenko, Sur les fonctions caractéristiques, Bull. l'Univ. Moscou, **1** (1937), P. Lévy, Théorie de l'addition des variables aléatoires, (1937), pp. 189–190.

2) T. Kawata, Non-vanishing of functions and related problems. Tohoku Math. Journ., **46** (1940), Theorems 4 and 11.