35. On an Extension of Bloch's Theorem.

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Though the following extension of Bloch's theorem is an immediate consequence of a general theorem due to L. Ahlfors[1], it will not be of no interest to prove it directly, by modifying Landau's proof[2] of Bloch's theorem.

Theorem. Let $w=f(z)=z+\cdots$ be meromorphic in $|z|\leqq 1$ and K be the Riemann sphere of diameter 1, which touches the w-plane at $w=0$ and F be the Riemann surface of the inverse function $\varphi(w)$ of $w=f(z)$, spread upon K. Then F contains a schlicht circle of radius $\geqq d>0$, where d is a numerical constant.

Proof. We put

$$\operatorname*{Max.}_{|z|\leq 1} \frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2)=M, \tag{1}$$

then, since $f(z)=z+\cdots$ is meromorphic in $|z|\leqq 1$, $1\leqq M<\infty$.

Let $\dfrac{|f'(z_0)|}{1+|f(z_0)|^2}(1-|z_0|^2)=M$, then $|z_0|<1$.

By $Z=\dfrac{z-z_0}{1-\bar{z}_0 z}$, $f(z)$ becomes $F(Z)$, so that

$$\frac{|F'(Z)|}{1+|F(Z)|^2}(1-|Z|^2)=\frac{|f'(z)|}{1+|f(z)|^2}(1-|z|^2)\leqq M, \tag{2}$$

$$\frac{F'(0)}{1+|F(0)|^2}=Me^{i\theta}. \tag{3}$$

We rotate K so that $F(0)=f(z_0)$ becomes $w=0$, then

$$G(Z)=e^{-i\theta}\frac{F(Z)-F(0)}{1+\bar{F}(0)F(Z)}=MZ+\cdots$$

is meromorphic in $|Z|\leqq 1$. Since

$$\frac{|G'(Z)|}{1+|G(Z)|^2}=\frac{|F'(Z)|}{1+|F(Z)|^2}\leqq\frac{M}{1-|Z|^2},$$

we have by integrating along a radius of $|Z|\leqq 1$,

$$\int_0^Z \frac{|G'(Z)|}{1+|G(Z)|^2}|dZ|\leqq\frac{M}{2}\log\frac{1+|Z|}{1-|Z|}. \tag{4}$$

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1) L. Ahlfors: Sur les domaines dans lesquels une fonction méromorphe prend des valeurs appartenant a une region donnée. Théorème VI. Acta Societatis Scientiarum Fennicae. Nova Series A. Tom II. (1933).

2) E. Landau: Über die Blochschen Konstante und zwei verwandte Weltkonstanten. Math. Z. **30** (1929).