110. On the Semi-ordered Ring and its Application to the Spectral Theorem.

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This note deals with some remarks about semi-ordered rings and their application to the spectral theorem. Semi-ordered rings have been treated jointly by Messrs. I. Vernikoff, S. Krein and A. Tovbin[1]. We first observe that for their result the assumption of the associative law of multiplication is unnecessary; it follows, as the commutativity, from the other axioms, and this fact will be of use in applications. Further, their theorems may be obtained rather easily also from Clifford-Lorenzen's theorem concerning semi-ordered abelian groups[2] by considering operator-domains. As for application, we prove a spectral theorem in the semi-ordered rings without appealing to the spectral theorem in vector lattice but relying upon Baire's category theorem. We thus obtain a new approach to the spectral theorem for bounded self-adjoint operators in a Hilbert space.

1. *Elementary observations about semi-ordered abelian groups.* Let G be a semi-ordered abelian group, that is, an abelian group which possesses a semi-order $x \geqq y$ (equivalent to $x-y \geqq 0$) such that

(i) if $x \geqq 0$ and $y \geqq 0$ then $x+y \geqq 0$,

(ii) if $x \geqq 0$ and $-x \geqq 0$ then $x=0$.

We assume a further condition:

(iii) if $nx \geqq 0$ for a certain natural number n, then $x \geqq 0$.

Let moreover G possess an Archimedean unit e:

(iv) $\begin{cases} \text{for any } x \text{ there exists a natural number } n=n(x) \text{ such that} \\ -ne \leqq x \leqq ne. \end{cases}$

And we call the totality N of those elements x in G satisfying $-e \leqq tx \leqq e$ (for every $t=1, 2, \dots$) the radical of G. N is a normal subgroup[3] of G, and the factor group G/N is also a semi-ordered group.

In virtue of the condition (iii) G is, according to Clifford-Lorenzen's

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1) Sur les anneaux semi-ordonnés, C. R. URSS, **30** (1941). Cf. also H. Nakano Teilweise geordnete Algebra, Jap. J. Math., **17** (1941).

2) A. H. Clifford: Partially ordered abelian groups, Ann. of Math., **41** (1940). P. Lorenzen: Abstracte Begründung der multiplikativen Idealtheorie, Math. Zeitschr., **45** (1939).

3) Here we call a subgroup of G normal when it is a kernel of an order-homomorphism of G. Thus a subgroup H is normal if and only if $x \in H$, $0 \leqq y \leqq x$ implies $y \in H$.