104. *A Note on Infinite Series.*

By Shin-ichi IZUMI and Gen-ichiro SUNOUCHI.
Mathematical Institute, Tohoku Imperial University, Sendai.
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Introduction. Theorem A. If the infinite series $\sum a_n$ *converges and* $a_n \downarrow 0$ *(a_n tends to zero monotonously), then* $na_n \to 0$.

This is the classical theorem due to Olivier. This is generalized by Cesàro[1], de la Vallée Poussin[2], Rademacher[3], Ostrowski[4], Knopp[5], Izumi[6] and Meyer-König[7]. Ostrowski's theorem reads as follows.

Theorem B. If $a_n \downarrow 0$, *then* $\sum a_n$ *converges when and only when* $s_n - na_n$ converges.

This theorem contains Olivier's theorem. On the other hand Cesàro proved that

Theorem C. Let p_n *and* q_n *be the number of positive and negative terms in*

$$s_n \equiv a_1 + a_2 + \cdots + a_n .$$

If $\sum a_n$ *converges and* $|a_n| \downarrow 0$, *then* $(p_n - q_n)a_n \to 0$.

These theorems suggest us the following theorem.

Theorem D. If $|a_n| \downarrow 0$, $\sum a_n$ *converges when and only when* $s_n - (p_n - q_n)|a_n|$ *converges as* $n \to \infty$.

But we can show that this is not true in general (§ 3). Therefore in order to get the theorem of this type, we need some additional conditions. We give two types of conditions. That is, the one is concerning the magnitude of a_n and the other is concerning the sign of a_n. This is given in Theorem 1 and 2. Conditions in the theorem are the best possible ones in a sense. Incidentally we give a new proof of all above theorems (§ 1). Finally we remark that our problem is transformed into that of function theory.

§ **1.** *Proof of Theorem C.* The identity

$$\begin{aligned}&(\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n)/\lambda_n \\ &\quad = s_n - \Big((\lambda_2 - \lambda_1)s_1 + (\lambda_3 - \lambda_2)s_2 + \cdots + (\lambda_n - \lambda_{n-1})s_{n-1}\Big)/\lambda_n \qquad (1)\end{aligned}$$

is well known and is easy to verify. Put $a_\nu \equiv |a_\nu| \cdot e_\nu$, $\lambda_\nu = 1/|a_\nu|$, then the left hand side of (1) becomes

$$(e_1 + e_2 + \cdots e_n)|a_n| = (p_n - q_n)|a_n| . \qquad (2)$$

Since $\lambda_n \uparrow \infty$, by the Toeplitz theorem

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1) Cesàro, Rom. Acc. Lincei Rend., **4** (1888), p. 133.
2) de la Vallée-Poussin, Cours d'Analyse Infinitesimale, **1** (1914), p. 408.
3) Rademacher, Math. Zeitschr., **11** (1921), p. 276.
4) Ostrowski, Jahresb. der D.M.V., **34** (1926), p. 161.
5) Knopp, Jahresb. der D.M.V., **37** (1928), p. 325.
6) Izumi, Proc. of the Physico-Math. Soc., **16** (1934), p. 127.
7) Meyer-König, Math. Zeitschr., **45** (1939), p. 751.