17. *Normed Rings and Spectral Theorems, III.*

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§ 1. *A Spectral theorem.* Let $\boldsymbol{R}$ be a function ring of real-valued continuous functions $S(M)$ on a bicompact Hausdorff space $\mathfrak{M}$. We assume that $\boldsymbol{R}$ satisfies

(1) for any pair $M_1, M_2 \in \mathfrak{M}$ there exists $S(M) \in \boldsymbol{R}$ such that $S(M_1) \neq S(M_2)$.

Then, by Gelfand-Silov's abstraction of Weierstrass' polynomial approximation theorem[1],

(2) every continuous function on $\mathfrak{M}$ may be uniformly approximated by functions $\in \boldsymbol{R}$.

Next let $F(S)$ be a linear functional on $\boldsymbol{R}$:

$$(3)\quad \begin{cases} F(\alpha S+\beta U)=\alpha F(S)+\beta F(U) \quad (\alpha, \beta=\text{scalars}), \\ F(S_n)\to F(S) \ (n\to\infty) \ \text{ if } \ \sup\limits_M |S_n(M)-S(M)| \to 0 \ (n\to\infty). \end{cases}$$

We further assume that

$$(4)\qquad F(S) \geqq 0 \quad \text{if} \quad S(M) \geqq 0 \quad \text{on } \mathfrak{M},$$

$$(5)\qquad F(I)=1, \quad \text{where} \quad I(M)\equiv 1 \quad \text{on } \mathfrak{M}.$$

Then, by (2)–(5) and Riesz-Markoff-Kakutani's theorem[2], we have the representation :

$$(6)\qquad F(S)=\int_{\mathfrak{M}} S(M)\varphi(dM) \qquad S\in\boldsymbol{R},$$

where φ is a non-negative, continuous from above set function countably additive on Borel sets $\subseteqq \mathfrak{M}$ and $\varphi(\mathfrak{M})=1$. Here the continuity from above means that the value of the function on a set is equal to the infimum of its values on open sets covering this set.

We have, from (6),

$$(7)\quad \begin{cases} F(T)=\int_{\mathfrak{M}} T(M)\varphi(dM)=\int_{\lambda_0}^{\lambda_1}\lambda d\tau(\lambda), \ \ \lambda_0=\inf\limits_M T(M), \ \ \lambda_1=\sup\limits_M T(M), \\ \tau(\lambda)=\varphi(M;\ T(M)<\lambda). \end{cases}$$

Put

$$(8)\qquad \mu=\sup_\lambda\big(\lambda\,;\ \tau(\lambda+\varepsilon)-\tau(\lambda-\varepsilon)>0 \quad \text{for all} \quad \varepsilon>0\big).$$

μ may be called as *the maximal spectrum of F referring to T.*

We will prove the

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1) Rec. Math., **9** (1941), Cf. also H. Nakano: 全國紙上數學談話會, **218** (1941).

2) A. Markoff: Rec. Math., **4** (1938). S. Kakutani: Proc. **19** (1943).