28. *A Kinematic Theory of Turbulence**.

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1. *Generalities.* In the theory of turbulence[1] the deviation of the velocity from its mean may be considered as a system of random vectors $u_\lambda(t, \mathfrak{X}, \omega)$, $\lambda=1,2,3$, where $t(\in R)$ is the time parameter and $\mathfrak{X}(\in R^3)$ denotes the position and $\omega\big(\in(\Omega, P)\big)$ is the elementary event. Then we have

$$\text{(1)} \qquad \mathcal{E}_\omega\big(u_\lambda(,\mathfrak{X},\omega)\big)=0\,.$$

When the system $\{u_\lambda(t,\mathfrak{X},\omega)\}$ is of Gaussian type[2], we say that the turbulence is of Gaussian type.

Now we define the moment tensor of the turbulence by

$$\text{(2)} \qquad R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})= \mathcal{E}_\omega\{u_\lambda(t, \mathfrak{X}, \omega)u_\mu(s, \mathfrak{Y}, \omega)\}\,.$$

Then $R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})$ is a positive-definite function of $(\lambda, t, \mathfrak{X})$ and $(\mu, s, \mathfrak{Y})$ in the sense of Bochner, namely we have

$$\text{(3)} \qquad R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})=R_{\mu\lambda}(s, \mathfrak{Y}\,;\, t, \mathfrak{X}) \quad \text{and}$$

$$\text{(4)} \qquad \textstyle\sum_{ij}\xi_i\xi_j R_{\lambda_i\lambda_j}(t_i, \mathfrak{X}_i\,;\, t_j, \mathfrak{X}_j)\geqq 0\,;$$

in fact (3) is evident by (2) and the left side of (4) is equal to $\mathcal{E}_\omega\Big\{\big(\sum_i\xi_i u_{\lambda_i}(t_i, \mathfrak{X}_i, \omega)\big)^2\Big\}$. Conversely the function $R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})$ satisfying (3) and (4) may be considered as the moment tensor of a turbulence of Gaussian type[3].

A turbulence is defined as temporally homogeneous, if its moment tensor satisfies

$$\text{(5)} \qquad R_{\lambda\mu}(t+\tau, \mathfrak{X}\,;\, s+\tau, \mathfrak{Y})=R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})\,.$$

It is defined as spatially homogeneous, if we have

$$\text{(6)} \qquad R_{\lambda\mu}(t, \mathfrak{X}+\mathfrak{a}\,;\, s, \mathfrak{Y}+\mathfrak{a})=R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})\,.$$

We say that it is isotopic if we have always

$$\text{(7)} \qquad \sum_{\lambda'\mu'} k_{\lambda'\lambda}k_{\mu'\mu}R_{\lambda'\mu'}\big(t, \mathfrak{X}\,;\, s, \mathfrak{X}+K(\mathfrak{Y}+\mathfrak{X})\big)=R_{\lambda\mu}(t, \mathfrak{X}\,;\, s, \mathfrak{Y})$$

for any orthogonal transformation $K\equiv\{k_{\lambda\mu}\,;\, \lambda, \mu=1,2,3\}$. We can easily prove by (3) that the isotropism implies the homogenuity.

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1) H. P. Robertson: The invariant theory of isotropic turbulence, Proc. Cambr. Phil. Soc. 36, 1940.

2) Cf. K. Itô: ガウス型確率變數系ニツイテ (全國紙上數學談話會第 261 號).

3) See Theorem 3 in my above-cited note (2).