43. On the Normal Stationary Process with no Hysteresis.

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§ 1. Let (Ω, P) be a probability field and $\mathfrak{M}$ a system of real valued random variables. $\mathfrak{M}$ is called to be *normal* or *of the Gaussian type*, if, for any $x_i(\omega) \in \mathfrak{M}$, $i=1, 2, \dots, n$, the random variable $\big(x_1(\omega), x_2(\omega), \dots, x_n(\omega)\big)$ is subjected to an n-dimensional (sometimes perhaps degenerated) Gaussian distribution. *This condition is equivalent to the property that, for any $x_i(\omega) \in \mathfrak{M}$, and for any real a_i, $\sum_{i=1}^{n} a_i x_i(\omega)$ is normally distributed.* Let $x_i(\omega)$, $i=1, 2, \dots, m$, $y_j(\omega)$, $j=1, 2, \dots, n$, be elements in a normal system $\mathfrak{M}$. *Then the non-correlatedness of $x_i(\omega)$ and $y_j(\omega)$ for any, i, j, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, implies the independence of $\big(x_1(\omega), x_2(\omega), \dots, x_m(\omega)\big)$ and $\big(y_1(\omega), y_2(\omega), \dots, y_n(\omega)\big)$.*

Let $x(t, \omega)$, $-\infty < t < \infty$, be a stochastic process. If the system of $x(t, \omega)$, $-\infty < t < \infty$, is normal, then the process will be said to be *normal*. If the (conditional) probability law of $x(t, \omega)$ under the condition that $x(t_1, \omega), x(t_2, \omega), \dots, x(t_n, \omega)$ should be given depends only on the value $x(t_n, \omega)$ for any $t_1 < t_2 < \cdots < t_n < t$, we say that $x(t, \omega)$ *has no hysteresis* or *is a simple Markoff process*. This terminology is applied to the case of a stochastic sequence $x(k, \omega)$, $k=0, \pm 1, \pm 2, \dots$.

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§ 2. *The form of the correlation function.*

Theorem 1. Let $x(k, \omega)$ be a normal stationary (in the sense of A. Khintchine) sequence. A necessary and sufficient condition that $x(k, \omega)$ should have no hysteresis is that its correlation function $\rho(k)$ is of the form α^k, $-1 \leqq \alpha \leqq 1$.

Proof. In order to avoid trivial complications we assume $E_\omega\big(x(k, \omega)\big)=0$, and $E_\omega\big(x(k, \omega)^2\big)=1$. If we define $\big(f(\omega), g(\omega)\big)$ by $E_\omega\big(f(\omega)g(\omega)\big)$, the closed linear subspace determined by the set $x(k, \omega)$, $k=0, \pm 1, \pm 2, \dots$, is considered as a Hilbert space, where *orthogonality implies (stochastic) independence* (Cf. § 1).

Sufficiency. For the proof it is sufficient to show that the conditional probability law of $x(k, \omega)$ under the condition that $x(k-i, \omega) = \xi_i$, $i=1, 2, \dots, n$, depends only on ξ_1.

We put $y(k, \omega) = x(k, \omega) - \alpha x(k-1, \omega)$. Then we have $E_\omega\big(y(k, \omega)x(k-i, \omega)\big) = \alpha^i - \alpha\alpha^{i-1} = 0$, $i=1, 2, \dots$. Since the sequence is normal, $y(k, \omega)$ is independent of $\big(x(k-1, \omega), x(k-2, \omega), \dots, x(k-n, \omega)\big)$. Therefore the probability law of $y(k, \omega)$ is invariant even if we add the condition: $x(k-i, \omega) = \xi_i$, $i=1, 2, \dots, n$. Therefore the probability