PAPERS COMMUNICATED

53. *On the Strong Summability of Series of Orthogonal Functions.*

By Gen-ichirô SUNOUCHI.

Mathematical Institute, Tohoku Imperial University, Sendai.

(Comm. by M. FUJIWARA, M.I.A., May 12, 1944.)

Let $\{\varphi_n(x)\}$ be a system of normalized orthogonal functions in (a, b) and consider the series

$$(1) \qquad \sum_{\nu=0}^{\infty} c_\nu \varphi_\nu(x)$$

such that

$$\sum_{\nu=0}^{\infty} c_\nu^2 < \infty .$$

By the Riesz-Fisher theorem, the series (1) converges in the mean to a function $f(x)$ in L^2. As usual we denote by $s_n(x)$ and $\sigma_n(x)$ the partial sum and $(C, 1)$-mean of the series (1) respectively. In this paper we discuss the convergency of

$$(2) \qquad \sum_{n=1}^{\infty} | s_n(x) - f(x) |^k/n , \qquad k > 1 ,$$

and

$$(3) \qquad \sum_{n=1}^{\infty} | s_n(x) - \sigma_n(x) |^k/n , \qquad k > 1 .$$

For the case of trigonometrical system, the former is considered by Hardy and Littlewood[1] and the latter by Zygmund[2].

As an application of our theory, we shall give an alternative proof of the Rademacher[3]-Menchof[4] theorem regarding the almost everywhere convergence of the series (1).

1. *Convergency of the series* $\sum_{n=1}^{\infty} (s_n - f)^2/n$.

(1.1) *In the series* (1), *we get*

$$\int_a^b \{\sum_{n=1}^{\infty} (s_n - f)^2/n\}\, dx \leqq A \sum_{n=1}^{\infty} c_n^2 \log n^{5)} .$$

For,

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b (s_n - f)^2 dx = \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{\nu=n+1}^{\infty} c_\nu^2 \right) = \sum_{\nu=2}^{\infty} c_\nu^2 \sum_{n=1}^{\nu-1} \frac{1}{n} \leqq A \sum_{\nu=1}^{\infty} c_\nu^2 \log \nu ,$$

which is the required.

For the case of trigonometrical system, we have

$$\sum_{n=1}^{\infty} c_n^2 \log n \sim \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |f(x+t) + f(x-t) - 2f(x)|^2/2t\, dx dt .$$

1) G. H. Hardy and J. E. Littlewood, Duke Math. Journ., **2** (1936), pp. 354–382.
2) A. Zygmund, Fund. Math., **30** (1938), pp. 170–196.
3) H. Rademacher, Math. Ann., **87** (1922), pp. 112–138.
4) D. Menchof, Fund. Math., **4** (1923), pp. 82–105.
5) $A, B, \ldots$ are constants, not always the same from one occurrence to another.