112. On Fourier Constants.

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G. H. Hardy[1] proved the following theorem:

(*A*) *If* $\{a_n\}$ *are the Fourier constants of a function of* L_p $(p \geqq 1)$, *then* $\{(\sum_{k=1}^{n} a_k)/n\}$ *are also the Fourier constants of a function of* L_p.

Recently T. Kawata[2] has proved a dual form of (*A*), that is:

(*B*) *If* $\{a_n\}$ *are the Fourier sine constants of a function of* L_p $(p > 1)$, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are also the Fourier sine constants of a function of* L_p. *Moreover if* $\{a_n\}$ *are the Fourier sine constants of a function of* L_z, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are the Fourier sine constants of a function of* L.

In the present note the author considers the case of cosine constants and completes (*B*) in the following form.

Theorem 1. If $\{a_n\}$ *are the Fourier constants of a function* L_p $(p > 1)$, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are also the Fourier constants of a function of* L_p. *Moreover if* $\{a_n\}$ *are the Fourier constants of a function of* L_z, *then* $\{\sum_{k=n}^{\infty} a_k/k\}$ *are the Fourier constants of a function of* L.

The method of proof is analogous to that of Kawata, but is somewhat delicate.

Proof of the case L_p. It is sufficient to prove the theorem for pure cosine series without constant term, that is $\int_0^{\pi} f(x)dx = 0$.

Let

$$(1) \qquad f(x) \sim \sum_{k=1}^{\infty} a_k \cos kx, \qquad f(x) \in L_p,$$

$$(2) \qquad g(x) \sim \sum_{k=1}^{\infty} \frac{1}{k} \cos kx,$$

then $g(x) \in L_r$ for all $r \geqq 1$ by the Hausdorff-Young theorem.

By Parseval's relation[3], we have

$$(3) \qquad \sum_{k=n}^{\infty} \frac{a_k}{k} = \frac{2}{\pi}\int_0^{\pi} f(x)g(x)dx - \frac{2}{\pi}\int_0^{\pi} f(x) \sum_{k=1}^{n-1} \frac{\cos kx}{k}\, dx.$$

The left-hand side series is summable $(C, 1)$, and further in this case it converges as $f(x) \in L_p$.

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1) G. H. Hardy, Messenger of Math., **58** (1928), 50–52.
2) T. Kawata, Proc. **20** (1944), 218–222.
3) A. Zygmund, *Trigonometrical series*, (1935), 88.