132. *On Simultaneous Extension of Continuous Functions.*

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1. Let Ω be a compact (=bicompact) Hausdorff space, and F a closed subset of Ω. Let $C(\Omega)$ be the normed ring of all complex-valued continuous functions on Ω, and let $C(F)$ be analogously defined. For any $x \in C(\Omega)$ and $x' \in C(F)$, their norms are defined by $\|x\|_\Omega = \max_{\omega \in \Omega} |x(\omega)|$ and $\|x'\|_F = \max_{\omega' \in F} |x'(\omega')|$, respectively.

Then, by Urysohn's extension theorem, to any $x' \in C(F)$ there corresponds an $x \in C(\Omega)$ such that $x'(\omega') = x(\omega')$ for any $\omega' \in F$. (When F is an essential subset of Ω, $x(\omega)$ is, of course, not unique). Thus a mapping $x = \varphi(x')$ of $C(F)$ into $C(\Omega)$ is defined. The purpose of this paper is to prove that we can take as φ a *linear* $\big(\varphi(x'+y') = \varphi(x') + \varphi(y')$ and $\varphi(\alpha x') = \alpha\varphi(x')$, where α is a complex number$\big)$, *multiplicative* $\big(\varphi(x'y') = \varphi(x')\varphi(y')\big)$ and *isometric* $\big(\|\varphi(x') - \varphi(y')\|_\Omega = \|x'-y'\|_F\big)$ mapping, if and only if F is a *retract* of Ω in the sense of K. Borsuk[1], i.e. if there exists a continuous mapping $\omega' = f(\omega)$ of Ω onto F such that $f(\omega') = \omega'$ on F.

2. *Lemma.*[2] *Let R be a closed subring*[3] *of $C(\Omega)$ containing the unit element of $C(\Omega)$ and satisfying the following condition:*

(*) $\qquad x(\omega) \in R$ *implies* $\overline{x(\omega)} \in R$.[4]

Then R is equivalent[5] *to $C(\Omega^*)$, where Ω^* is a certain continuous image of Ω. Conversely, if Ω^* is a continuous image of Ω, then $C(\Omega^*)$ is equivalent to some closed subring of $C(\Omega)$ which contains the unit of $C(\Omega)$ and satisfies the condition* (*).

We shall sketch the proof: To any maximal ideal[6] of R there corresponds at least one point of Ω and to any point of Ω there corresponds one maximal ideal of R. From this follows easily that the set Ω^* of all maximal ideals of R, which is topologized by the weak topology, is a continuous image of $\Omega : \Omega^* = g(\Omega)$. Then R is equivalent to $C(\Omega^*)$ by the correspondence $x(\omega) \to x^*(\omega^*)$, where $x^*\big(g(\omega)\big) = x(\omega)$.

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1) K. Borsuk, Sur les rétractes, Fund. Math., **17** (1931), 152–170.

2) G. Šilov, Ideals and subrings of the ring of continuous functions, C. R. URSS, **22** (1939), 7–10.

3) If not mentioned explicitly, we do not assume that a subring of R contains the unit element of R.

4) $\bar{z}$ denotes the conjugate complex number of z.

5) Two normed rings are *equivalent* if there exists an isometric isomorphism between them.

6) Concerning these notions, see I. Gelfand and A. Kolmogoroff, On rings of continuous functions on topological spaces, C. R. URSS, **22** (1939), 11–15, and I. Gelfand and G. Šilov, Über verschiedene Methoden der Einführung der Topologie in die Menge der maximalen Ideale eines normierten Ringes, Recueil Math., **9** (1941), 25–39.