131. On Brownian Motions in n-Space.

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1. The *one-dimensional Brownian motion* $\{x(t, \omega) \mid -\infty < t < \infty, \omega \in \Omega\}$ is defined as a real-valued temporally homogeneous differential process with no moving discontinuities and having a Gaussian distribution :[1]

$$(1) \qquad Pr\{\omega \mid a < x(t, \omega) - x(s, \omega) < b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{u^2}{2(t-s)}} du,$$

where $-\infty < s < t < \infty$ and $-\infty \leqq a < b \leqq \infty$. The *n-dimensional Brownian motion* (or equivalently, the *Brownian motion in n-space* R^n) $\{\boldsymbol{x}(t, \omega) = \{x_i(t, \omega),\ i = 1, \dots, n\} \mid -\infty < t < \infty,\ \omega \in \Omega\}$ is an n-system of mutually independent one-dimensional Brownian motions $x_i(t, \omega)$ (with the same normalization). It is known that this definition is independent of the choice of coordinate system in R^n.

The mathematical theory of Brownian motions was discussed by N. Wiener[2] and P. Lévy[3], and many important results were obtained. But, for the most part, their investigations were restricted to the one- or the two-dimensional case or concerned only with the properties of Brownian motions in which the dimension number n does not play an important rôle. The purpose of this paper is to discuss some new properties of Brownian motions which do not appear in the one- or the two-dimensional case. Our main results are stated in Theorems 1, 2 and 4.

It is to be noticed that in the Brownian motion in 3-space almost all paths constitute a nowhere dense set in R^3 (Theorem 4) and tend to ∞ as $t \to \infty$ (Theorem 2), while, as may be shown[4] by appealing to the theory of harmonic functions, in the two-dimensional Brownian motion almost all paths describe a curve everywhere dense in the entire plane and come back to any neighborhood of any given point infinitely many times (for infinitely large value of t). It is known that in the two-dimensional Brownian motion almost all paths have infinitely many double points. We have a conjecture that already in 3-space almost all paths have no double points, although thus far we could prove this only for 5-space (Theorem 1).

2. *Lemma 1*[5]. *If $x(t, \omega)$ is a one-dimensional Brownian motion and if $-\infty < t_0 < t_1 < \infty$, then*

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1) $Pr\{\omega|A\}$ denotes the probability (=measure) of the set of all $\omega \in \Omega$ with the property A, i.e. the probability of A.

2) N. Wiener, Generalized harmonic analysis, Acta Math., **54** (1932).

3) P. Lévy, Les mouvements browniens plans, Amer. Journ. of Math., **62** (1940).

4) This will be discussed in a forthcoming paper of the author.

5) Cf. P. Lévy, loc. cit. 3).