Cauchy-Kovalevskaja-Nagumo type theorems for PDEs with shrinkings

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(Communicated by Heisuke Hironaka, M.J.A., Dec. 13, 1999)

1. Introduction. By the Cauchy-Kovalevskaja theorem we know that, if $f(t,x,u,v)$ is an analytic function of $(t,x,u,v) \in \mathbf{R}^4$ then the Cauchy problem

$$(1.1) \qquad \partial_1 u(t,x) = f(t,x,u(t,x),\partial_2 u(t,x)),$$

$$(1.2) \qquad u(0,x) = 0,$$

where ∂_i is the partial differentiation in the ith variable, has a unique analytic local solution. But it seems impossible simply to replace the first order derivative $\partial_2 u(t,x)$ in the equation (1.1) by a higher order derivative $\partial_2^p u(t,x)$ with $p > 1$.

So it was quite surprising to us to learn that Augustynowicz *et al.* [2], [3] had solved the Cauchy problem for an equation of the form

$$(1.3) \quad \partial_1 u(t,x) = a(t,x)\partial_2^p u(t,\alpha(t,x)x) + g(t,x)$$

or

$$(1.4) \quad \partial_1 u(t,x) = a(t,x)\partial_2^p u(\alpha(t,x)t,x) + g(t,x).$$

In (1.3) and (1.4) it is assumed that $a(t,x)$, $g(t,x)$ and $\alpha(t,x)$ are given analytic functions of (t,x). The function α is called a *delay* and has the property that $0 < \alpha(t,x) < 1$ if $|t| + |x|$ is small. It seems that a *delay* plays the role of appeasing the disturbance caused by differentiation in x.

Our purpose here is to generalize the above mentioned results in [2], [3] as much as possible. We want to consider non-linear differential equations instead of linear ones such as (1.3) or (1.4). Also we want mainly to consider the case where the differential equation is not analytic in t.

As the first step toward these ends we consider in this note two simple PDEs of the form

$$(1.5) \quad \partial_1 u(t,x) = f(t,x,u(t,x),\partial_2^p u(t,\alpha(t,x)x))$$

and

$$(1.6) \quad \partial_1 u(t,x) = f(t,x,u(t,x),\partial_2^p u(\alpha(t)t,x)).$$

In (1.5) and (1.6) $\alpha(t,x)$ or $\alpha(t)$ is a function with properties similar to those of $\alpha(t,x)$ in (1.3) and (1.4). We call $\alpha(t,x)$ or $\alpha(t)$ a *shrinking* instead of a *delay*, since it may sound strange to use the word *delay* for the space variable. Our result for the equation (1.5) will be stated in §2 as Theorem 2.1. Our result for the equation for (1.6) will be stated in §3 as Theorem 3.2. Note that the discussion for the equation (1.5) is much simpler than that for the equation (1.6). For instance, the domain of existence of a solution to the Cauchy problem (1.6)-(1.2) is of the strange form $\{(t,x)\,;\, |t| < a,\ R - |x| - |t|^{1/(p+1)} > 0\}$, while the corresponding domain for the problem (1.5)-(1.2) is of the simple form $\{(t,x)\,;\, |t| < a,\ |x| < R\}$. In order to obtain the result of §3 we use a variant of the famous trick in Nagumo [1].

For the reason why $\alpha(t)$ in (1.6) cannot be replaced by $\alpha(t,x)$ and further possibilities of obtaining other related results see the remarks in §4.

2. Shrinking in the space variable. The purpose of this section is to solve the Cauchy problem for the equation (1.5). Our result will be stated in the Theorem 2.1 below. In the theorem we use the following notation. If T, R and S are positive constants, we write

$$A(T,R) = \{(t,x) \in \mathbf{R} \times \mathbf{C}\,;\, |t| < T,\ |x| < R\},$$
$$B(T,R,S) = \{(t,x,u,v) \in \mathbf{R} \times \mathbf{C}^3\,;\ (t,x) \in A(T,R),\ |u| < S,\ |v| < S\}.$$

Theorem 2.1. *Let T, R, S and m be positive constant. Assume that $m < 1$. Let p be an integer $\geq$ 1. In the differential equation* (1.5) *assume that*

(i) *$f(t,x,u,v)$ is a complex valued bounded continuous function of $(t,x,u,v) \in B(T,R,S)$;*
(ii) *$f(t,x,u,v)$ is analytic in (x,u,v);*
(iii) *$\alpha(t,x)$ is a complex valued continuous function of $(t,x) \in A(T,R)$;*
(iv) *$\alpha(t,x)$ is analytic in x and satisfies the inequality*

$$|\alpha(t,x)| \leq m.$$

Then there is a positive constant a such that the Cauchy problem (1.5)-(1.2) *has a unique solution*