# A generalization of the Hardy theorem to semisimple Lie groups

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**1. Introduction.** The classical uncertainty principle asserts that a function and its Fourier transform cannot both be concentrated on intervals of small measure. In the case of the Euclidean space, various forms of the uncertainty principle are known. One of them is the following Hardy theorem (cf. [1, pp. 155-158]). If a measurable function $f$ on $\mathbf{R}$ satisfies $|f| \leq C\exp\{-ax^2\}$ and $|\hat{f}| \leq C\exp\{-by^2\}$ and $ab > 1/4$, then $f = 0$ (a.e.). Here we take $\hat{f}(y) = (1/\sqrt{2\pi})\int_{-\infty}^{\infty} f(x)\exp\{\sqrt{-1}xy\}dx$ as the definition of the Fourier transform of $f$.

Recently A. Sitaram and M. Sundari [6] generalized this theorem to the cases of the semisimple Lie groups with one conjugacy class of Cartan subgroups, the Riemannian symmetric spaces and $SL(2,\mathbf{R})$. On the other hand M. Sundari [7] showed the Hardy theorem for the Euclidean motion group. And also M. Eguchi, S. Koizumi and K. Kumahara generalized the results to Cartan motion group [2] and gave an $L^p$ version for motion groups [3]. In this paper we give an analogue of the Hardy theorem to noncompact semisimple Lie groups.

**2. Notation and preliminaries.** The standard symbols $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ shall be used for the set of the integers, the real numbers and the complex numbers. If $V$ is a vector space over $\mathbf{R}$, $V_c$, $V^*$ and $V_c^*$ denote its complexification, its real dual and its complex dual, respectively. For a Lie group $L$, $\hat{L}$ denotes the set of all equivalence classes of irreducible unitary representations of $L$. If $L$ is a reductive Lie group, $\hat{L}_{\rm disc}$ denotes the subset comprised of all equivalence classes of discrete series. As usual for a Lie group, we use lower case German letters to denote its Lie algebras. If $\mathcal{H}$ is a complex separable Hilbert space, the operator norm on $\mathcal{H}$ will be denoted by $\|\cdot\|_\infty$.

Let $G$ be a connected semisimple Lie group with finite center. We fix a maximal compact subgroup $K$ of $G$ and denote by $\theta$ the corresponding Cartan involution. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ defined by $\theta$. Let $\mathfrak{a}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and denote by $\Sigma$ the set of all restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$. We fix an order in $\mathfrak{a}_\mathfrak{p}^*$ and denote by $\Sigma^+$ the set of all positive restricted roots. Let $\{\alpha_1, \cdots, \alpha_l\}$ be the set of simple roots of $(\mathfrak{g}, \mathfrak{a}_\mathfrak{p})$ and put $L = \left\{\sum_{i=1}^l n_i\alpha_i \; ; \; n_i \in \mathbf{Z} \quad (i = 1, \cdots, l)\right\}$ and $L^+ = \left\{\sum_{i=1}^l n_i\alpha_i \; ; \; n_i \in \mathbf{Z}_{\geq 0} \quad (i = 1, \cdots, l)\right\}$. For each $\lambda = \sum_{i=1}^l \lambda_i\alpha_i \in L^+$, put $|\lambda| = \sum_{i=1}^l \lambda_i$. For $q = (q_1, \ldots, q_l) \in \mathbf{Z}_{\geq 0}^l$, we also put $\alpha(H)^q = \alpha_1(H)^{q_1}\cdots\alpha_l(H)^{q_l} \quad (H \in \mathfrak{a})$ and $|q| = \sum_{i=1}^l q_i$. Let $\mathfrak{a}_\mathfrak{p}^-$ be a choice of negative Weyl chamber. We set $\mathfrak{n}_\mathfrak{p} = \sum_{\lambda\in\Sigma^+}\mathfrak{g}_\lambda$ and put $A_\mathfrak{p} = \exp\mathfrak{a}_\mathfrak{p}$, $N_\mathfrak{p} = \exp\mathfrak{n}_\mathfrak{p}$ and $M_\mathfrak{p} = Z_K(\mathfrak{a}_\mathfrak{p})$. Then $P_0 = M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$ is a minimal parabolic subgroup of $G$. When $g = k\exp X \quad (k \in K, X \in \mathfrak{p})$, we set $\sigma(g) = \|X\|$, where $\|\cdot\|$ denoting the norm coming from the Killing form of $\mathfrak{g}$.

We write $\mathrm{Car}(G)$ for the set of all $\theta$-stable Cartan subgroups and denote by $\mathrm{Car}'(G)$ the subset of $\mathrm{Car}(G)$ comprised of all noncompact ones. For $J \in \mathrm{Car}(G)$, let $P_J = M_JA_JN_J$ be the Langlands decomposition of the cuspidal parabolic subgroup $P_J$ associated to $J$. We remark that if $J \in \mathrm{Car}'(G)$ then $A_J \neq \{e\}$.

Under the decomposition $G{=}KP_J{=}KM_JA_JN_J$, each $g{\in}G$ can be written as $g{=}\kappa(g)\mu_J(g)\exp H_J(g)$ $n_J(g)$, where $\kappa(g) \in K$, $\mu_J(g) \in M_J$, $H_J(g) \in \mathfrak{a}_J$ and $n_J(g) \in N_J$. Let $\sigma \in (\hat{M_J})_{\rm disc}$ and $\nu \in \mathfrak{a}_J^*$. We denote by $(\pi_{J,\sigma,\nu}, \mathcal{H}^{J,\sigma,\nu})$ the representation induced from $\sigma\otimes\nu\otimes 1$ of $P_J$ to $G$. Then it is known that $(\pi_{J,\sigma,\nu}, \mathcal{H}^{J,\sigma,\nu})$ is unitary.

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