Class number two problem for real quadratic fields with fundamental units with the positive norm

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1. Introduction and notations. Throughout this paper, we denote by N the set of positive rational integers, and put $N_0 = N \cup \{0\}$. Z will mean as usual the set of rational integers. For a square-free $D \in N$, the real quadratic field $Q(\sqrt{D})$ will be denoted by k, its class number by h_k and its fundamental unit > 1 by $\varepsilon_D = (t + u\sqrt{D})/2$. The norm map from k to Q will be denoted by N. The class number two problem requires to determine the set of all D for which $h_k = 2$ under certain conditions. This problem was solved by *Katayama* [2,3] with one possible exception for the conditions $N\varepsilon_D = -1$, $1 \leq u \leq 200$; by *Mollin and Williams* [5] for k of *Extended Richaud-Degert* type (i.e. with $D = m^2 + r$ where $4m \equiv 0 \pmod{r}$), also with one possible exception; and by *Taya and Terai* [7] for k of *Narrow Richaud-Degert* type (i.e. with $r = \pm 1$ or ± 4).

In this paper, we shall consider this problem for the case $N\varepsilon_D = 1$, $1 \leq u \leq 100$ and solve it with one possible exception (see Theorem below).

2. Lemmas and propositions. We begin by citing two known results as Lemmas 1,2 (The letters N, D, ε_D, t, u will always keep the meanings explained above. For a real number x, $[x]$ means as usual the greatest integer $\leq x$).

Lemma 1 (*Yokoi* [11]). Suppose $N\varepsilon_D = 1$. Then the following conditions for $n, v \in N_0$, $w \in Z$ determine these numbers uniquely, and we have $n = [t/u^2]$, $w = D - 2tn + u^2n^2$:

$$t = u^2n + v,\ v^2 - 4 = wu^2,\ v < u^2$$
$$D = u^2n^2 + 2vn + w.$$

For our real quadratic field $k = Q(\sqrt{D})$, we denote by d_k its discriminant (i.e. d_k is D or $4D$ according as $D \equiv 1 \pmod 4$ or $\equiv 2, 3 \pmod 4$), by χ_k Kronecker character of k and by $L(1, \chi_k)$ the Dirichlet L-function with this character.

Lemma 2 (*Tatuzawa* [6]). Suppose $d_k \geq \max(e^{1/\alpha}, e^{11.2})$ for a real number α with $0 < \alpha < 1/2$. Then we have

$$L(1, \chi_k) > \frac{0.655\alpha}{d_k^\alpha}$$

with one possible exception of k.

The following lemma will be used immediately afterward:

Lemma 3. We have $\varepsilon_D < 2u\sqrt{D}$.

Proof. This follows easily from $t = \sqrt{Du^2 \pm 4} < u\sqrt{D} + 2$. Q. E. D.

Let D be a square-free number $\in N$ for which $N\varepsilon_D = 1$ and n, v, w be the numbers $\in Z$ determined by the conditions in Lemma 1. From Lemmas 2,3, we can deduce the following

Proposition 1. D, n, v, w being as above, there exists a real number $\upsilon(u)$ determined by u, such that $h_k > 2$ follows from $n \geq \upsilon(u)$, with one possible exception of D.

Proof. From Lemma 2 and the well-known Dirichlet's class number formula, we get

$$h_k = \frac{\sqrt{d_k}}{2\log\varepsilon_D} L(1, \chi_k) > \frac{0.655}{2\log\varepsilon_D} \frac{\sqrt{d_k}\, d_k^{-1/y}}{y}$$

for $y \geq 11.2$ and $d_k \geq e^y$, with one possible exception of k. Since $\varepsilon_D < 2u\sqrt{D} \leq 2u\sqrt{d_k}$ by Lemma 3, we have

$$h_k > \frac{0.655 d_k^{1/2-1/y}}{y(\log d_k + 2\log u + 2\log 2)}.$$

y being fixed, the right-hand side is a monotone increasing function of d_k. Thus we can replace here d_k by e^y to obtain

$$h_k > \frac{0.655 d_k^{y/2-1}}{y(y + 2\log u + 2\log 2)}$$

Let us denote by $f_u(y)$ the right-hand side of this inequality. For any fixed u, $f_u(y)$ tends to ∞ as $y \to \infty$. So there exists a real number $c(u) \geq 11.2$ satisfying $f_u(c(u)) \geq 2$. Thus, solving the inequality

$$e^{c(u)} \leq D = u^2n^2 + 2vn + w \leq d_k$$

for n, one can find a real number $\upsilon(u)$ such that $h_k > f_u(c(u)) \geq 2$ for $n \geq \upsilon(u)$. Q. E. D.

In fact, we may take $\upsilon(u) \geq \sqrt{4 + u^2e^{c(u)}} / u^2$. Moreover, we can choose $c(u) < 16.5$ for 1