On the λ-invariants of totally real fields

By Jangheon OH*)

KIAS, 207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea

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1. Introduction. Let k be a number field and p be a prime number, and let $k = k_0 \subset k_1 \subset \cdots \subset k_n \subset \cdots \subset k_\infty$ be the cyclotomic $\boldsymbol{Z}_p$-extension of k. We denote by $\mu_p(k)$, $\lambda_p(k)$ the Iwasawa invariants of the cyclotomic $\boldsymbol{Z}_p$-extension of k. It is well-known that $\mu_p(k)$ vanishes for any abelian number field k. Greenberg's conjecture claims that both $\mu_p(k)$ and $\lambda_p(k)$ are zero for any totally real number field k. In this paper, we shall prove the following

Theorem 1. *Let p and q be prime numbers such that $p \equiv 3 \bmod 8$, $q \equiv -1 \bmod 8$, $p \not\equiv 3 \bmod 16$, $q \not\equiv -1 \bmod 16$. Then the Iwasawa invariant $\lambda_2(\boldsymbol{Q}(\sqrt{pq}))$ is zero. Let p, q and r be prime numbers such that $p, q \equiv 3 \bmod 8$, $p, q \not\equiv 3 \bmod 16$. $r \equiv 1 \bmod 4$, $r \not\equiv 1 \bmod 8$. Then the Iwasawa invariant $\lambda_2(\boldsymbol{Q}(\sqrt{pqr}))$ is zero if there is no element α in the unit group of $k_1 = \boldsymbol{Q}(\sqrt{pqr}, \sqrt{2})$ such that $N_{k_1/Q_1}\alpha = -1$.*

Let p and ℓ be odd prime numbers such that $p \equiv 1 \bmod \ell$. Let k be a subfield of degree ℓ of $\boldsymbol{Q}(\zeta_{p\ell^2})$ in which p and ℓ ramify. Here $\zeta_{p\ell^2}$ is a primitive $p\ell^2$-th root of unity. We will prove the following

Theorem 2. *Let p and ℓ be odd prime numbers such that $p \equiv 1 \bmod \ell$, $p \not\equiv 1 \bmod \ell^2$. Then the Iwasawa invariants $\mu_\ell(k)$ and $\lambda_\ell(k)$ vanish, where k is the number field constructed above.*

Now let p be a prime number and k be a totally real number field and K be a real cyclic extension of degree p over k, which satisfies $K \cap k_\infty = k$. Let $S_{K_\infty/k_\infty} = \{w$: prime ideal of $K_\infty \mid w$ is prime to p and ramified in $K_\infty/k_\infty\}$.

In [1], Iwasawa proved a "plus-version" of Kida's formula. In [2], the following theorem is obtained by using the above Iwasawa's formula.

Theorem 3. *Let p be a prime number, k a totally real number field of finite degree and K a real cyclic extension of degree p over k. Assume that k_∞ has only one prime ideal lying over p and that the class number of k is not divisible by p. Then, the following are equivalent:*

(1) $\lambda_p(K) = 0$.

(2) *For any prime ideal w of K_∞ which is prime to p and ramified in K_∞/k_∞, the order of ideal class of w is prime to p.*

In this paper, we apply Theorem 3 to prove Theorem 1 and Theorem 2. We state another ingredient needed here. Let K be a cyclic extension of a number field F. Let $G = Gal(K/F)$. For each valuation v of F we let $e(v)$ be the ramification index of v in K/F. We put $e(K/F) = \Pi_v e(v)$. We let E_K denote the group of units, C_K the group of ideal classes, C_K^G the set of ambiguous ideal class groups, and C'^G_K the set of ideal class groups containing ambiguous ideal of K, respectively. We will use the following "genus formula":

Theorem 4. *Let K/F be a cyclic extension with Galois group G. Then*

$$(1) \quad |C_K^G| = \frac{h(F)e(K/F)}{[K:F](E_F : N_{K/F}K^* \cap E_F)}.$$

$$|C'^G_K| = \frac{h(F)e(K/F)}{[K:F](E_F : N_{K/F}E_K)}.$$

Proof. See [3, p. 307]. □

2. Proof of theorems. Before proving Theorem 1, we need the following

Lemma 1. *Let D be a square free positive integer such that there exists a prime number $q \mid D$ such that $q \equiv -1 \bmod 8$. Let $k = \boldsymbol{Q}(\sqrt{D})$. Then there is no element α in the first layer k_1 in the cyclotomic $\boldsymbol{Z}_2$-extension of k such that*

$$N_{k_1/Q_1}(\alpha) = -1.$$

Proof. First note that $(\frac{-1}{q}) = -1$ and $(\frac{2}{q}) = 1$. Suppose that there is an α in k_1 such that

$$(2) \qquad N_{k_1/Q_1}(\alpha) = -1.$$

Write $\alpha = x + y\sqrt{2} + z\sqrt{D} + w\sqrt{2D}$, where x, y, z and w are in $\boldsymbol{Q}$.

Then by (2) we have

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