Gröbner deformations of regular holonomic systems

By Mutsumi SAITO,*) Bernd STURMFELS,**) and Nobuki TAKAYAMA***)

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1. Torus-fixed ideals in the Weyl algebra. This is a research announcement of results in the first part of our monograph [15]. Let $D = \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ denote the Weyl algebra with complex coefficients. Thus D is the free associative $\mathbf{C}$-algebra on $2n$ generators modulo the relations $x_i x_j = x_j x_i$, $\partial_i \partial_j = \partial_j \partial_i$, $x_i \partial_j = \partial_j x_i - \delta_{ij}$. Left ideals in D are called *D-ideals*. They represent systems of linear partial differential equations with polynomial coefficients. The torus $(\mathbf{C}^*)^n$ acts on the Weyl algebra by $\partial_i \longmapsto t_i \partial_i$ and $x_i \longmapsto t_i^{-1} x_i$ for $(t_1, \ldots, t_n) \in (\mathbf{C}^*)^n$. We abbreviate $\theta_i = x_i \partial_i$. The set of elements in D which are fixed by $(\mathbf{C}^*)^n$ equals the commutative polynomial subring $\mathbf{C}[\theta] = \mathbf{C}[\theta_1, \ldots, \theta_n]$.

Lemma 1.1. *A D-ideal J is torus-fixed if and only if J is generated by (finitely many) elements of the form $x^a \cdot p(\theta) \cdot \partial^b$ where $a, b \in \mathbf{N}^n$ and $p(\theta) \in \mathbf{C}[\theta]$.*

Each $f \in D$ is written uniquely as a finite sum $f = \sum_{a,b \in \mathbf{N}^n} c_{ab} x^a \partial^b$ with $c_{ab} \in \mathbf{C}$. Fix $u, v \in \mathbf{R}^n$ with $u + v \geq 0$. Then $\mathrm{in}_{(u,v)}(f) \in D$ is the subsum of all terms $c_{ab} x^a \partial^b$ for which $u \cdot a + v \cdot b$ is maximal. For a D-ideal I we define the *initial ideal* $\mathrm{in}_{(u,v)}(I)$ to be the $\mathbf{C}$-vector space spanned by $\{\mathrm{in}_{(u,v)}(f) : f \in I\}$. If $u + v > 0$ then $\mathrm{in}_{(u,v)}(I)$ is generally not a D-ideal; it is an ideal in the commutative polynomial ring $gr(D) = \mathbf{C}[x, \xi] = \mathbf{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. Generators for the initial ideal can be computed by the Weyl algebra version of Buchberger's Gröbner basis algorithm; see e.g. [3] and [6] for early treatments and [13] for a precise introduction and recent applications.

If $u + v = 0$ then the initial ideal is a D-ideal. For $w \in \mathbf{R}^n$ we call $\mathrm{in}_{(-w,w)}(I)$ a *Gröbner deformation* of I. Specifically, if $w \in \mathbf{Z}^n$ then the D-ideal $\mathrm{in}_{(-w,w)}(I)$ is regarded as the limit of I under the one-parameter subgroup of $(\mathbf{C}^*)^n$ defined by w.

Lemma 1.2. *For generic $w \in \mathbf{R}^n$, the initial D-ideal $\mathrm{in}_{(-w,w)}(I)$ is torus-fixed.*

Let $D^{\pm} := \mathbf{C}\langle x_1^{\pm 1}, \ldots, x_n^{\pm 1}, \partial_1, \ldots, \partial_n\rangle$ be the ring of differential operators on $(\mathbf{C}^*)^n$. For a D-ideal I define the commutative polynomial ideal $\tilde{I} := D^{\pm} I \cap \mathbf{C}[\theta]$.

Proposition 1.3. *If J is a torus-fixed D-ideal then $\tilde{J} \subset \mathbf{C}[\theta]$ is generated by $p(\theta - b) \cdot \prod_{i=1}^{n} \prod_{j=1}^{b_i} (\theta_i + 1 - j)$ where $x^a \cdot p(\theta) \cdot \partial^b$ runs over a generating set of J.*

2. Holonomic rank under Gröbner deformations. Abbreviate $e := (1, 1, \ldots, 1) \in \mathbf{R}^n$. The ideal $\mathrm{in}_{(0,e)}(I)$ in $\mathbf{C}[x, \xi]$ is called the *characteristic ideal* of the D-ideal I. The *Fundamental Theorem of Algebraic Analysis* ([5],[12],[14]) states that each minimal prime of the characteristic ideal $\mathrm{in}_{(0,e)}(I)$ has dimension $\geq n$. If $\mathrm{in}_{(0,e)}(I)$ has dimension n then I is *holonomic*. In this case the following vector space dimension is finite and is called the *holonomic rank* of I:

(2.1) $rank(I) = \dim_{\mathbf{C}(x)}(\mathbf{C}(x)[\xi]/\mathbf{C}(x)[\xi] \cdot \mathrm{in}_{(0,e)}(I)).$

Here $\mathbf{C}(x) = \mathbf{C}(x_1, \ldots, x_n)$. The holonomic rank equals the dimension of the $\mathbf{C}$-vector space of holomorphic solutions to I at any point outside the singular locus.

Theorem 2.1. *Let I be a holonomic D-ideal and $w \in \mathbf{R}^n$. Then $\mathrm{in}_{(-w,w)}(I)$ is holonomic and*

(2.2) $rank(\mathrm{in}_{(-w,w)}(I)) \leq rank(I).$

Our proof of Theorem 2.1 is based on a walk in the *Gröbner fan* $GF(I)$ as defined in [1]. This fan decomposes the closed half space $\{u + v \geq 0\}$ of $\mathbf{R}^{2n}$ into finitely many convex polyhedral cones, one for each initial monomial ideal $\mathrm{in}_{(u,v)}(I) \subset \mathbf{C}[x, \xi]$.

Let $\mathfrak{D}$ be the sheaf of algebraic differential operators on $\mathbf{C}^n$. A holonomic D-ideal I is called

*) Department of Mathematics, Hokkaido University, Sapporo, 060-0810.
**) Department of Mathematics, University of California, Berkeley, CA 94720, U.S.A.; *and* Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502.
***) Department of Mathematics, Kobe University, Kobe, 657-8501.