Three facts of valuation theory

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1. Statement of results. Let k be a field and (R, m) a local integral k-algebra with field of fractions K. We study k-valuations ν of K with a center in R, that is, such that their valuation ring (R_ν, m_ν) contains R and $m_\nu \cap k = (0)$. We denote by Φ the totally ordered group of the valuation and set $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$. The valuation determines on R a filtration defined by the ideals

$$P_\phi(R) = \{x \in R / \nu(x) \geq \phi\}$$
$$\text{or } P_\phi^+(R) = \{x \in R / \nu(x) > \phi\},$$

and the associated graded ring introduced by Spivakovsky ([6], see also [4], [7]):

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Gamma} P_\phi / P_\phi^+,$$

which is a Γ (or Φ_+)-graded $(R/m_\nu \cap R)$-algebra. We assume throughout that Φ has finite rational rank $\mathrm{r}(\Phi)$ (and therefore is countable) and finite height (or rank) $\mathrm{h}(\Phi)$ The three facts, extracted from [8], are the following:

1) **A connexion between valuation theory and toric geometry**

Proposition 1.1. *For any specialization* (*see* [10],*vol.*2, *Chap.* VI,§16) *of the valuation ν to a valuation ν_0 with a center in R and such that $m_{\nu_0} \cap R = m$ and the residue field extension $k_R \to k_{\nu_0} = R_{\nu_0}/m_{\nu_0}$ induced by the inclusion $R \subset R_{\nu_0}$ is trivial, the algebra $\mathrm{gr}_{\nu 0} R$ is isomorphic to a quotient of a polynomial ring $k_R[(U_i)_{i \in I}]$ with coefficients in k_R and possibly countably many indeterminates by a binomial ideal, i.e. an ideal with* (*possibly countably many*) *generators of the form $U^m - \lambda_{mn} U^n$ where $U^m = U_1^{m_1} \cdots U_s^{ms}$ and $\lambda_{mn} \in k_R^*$. It means that it is a déformation of a* (*non normal*) *toric variety* (*see* [2]), *possibly of infinite embedding dimension, which is nothing but* $\mathrm{Spec} k_R[t^\Gamma]$, *where $k_R[t^\Gamma]$ is the semigroup algebra of Γ, obtained by replacing all λ_{mn} by* 1.

2) **Structure of valuation semigroup algebras and regularity of $\mathrm{gr}_\nu R_\nu$**

Proposition 1.2. *The graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ is a filtering direct limit of termic maps* (*i.e mapping a variable to a term, of the form constant a monomial*) *between polynomial subalgebras in* $\mathrm{r}(\Phi)$ *variables. The semigroup algebra $k_\nu[t^{\Phi+}]$ is the direct limit of the corresponding system of toric* (*or monomial*) *maps, obtained by replacing all the constants by* 1.

3) **Noetherianity of ν-adic completions**

Proposition 1.3. *Assume that R is an analytically irreducible noetherian local ring. If ν_1 denotes the valuation of height one with which ν is composed, and* $\mathbf{p} = m_{\nu 1} \cap R$ *the center of ν_1 on R, then the completion $\hat{R}^\nu$ of the ring R with respect to the topology defined by the $(P_\phi)_{\phi \in \Phi_+}$-filtration, is isomorphic as topological ring to a quotient of the* $\mathbf{p}$-*adic completion $\hat{R}^{\mathbf{p}}$ of R; it is noetherian.*

In particular, if R is excellent, so is $\hat{R}^\nu$ since $\hat{R}^{\mathbf{p}}$ is excellent by ([5]).

2. Ideas of proofs. 1) Since R_ν is a valuation ring, the Φ_+-graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ has the property that each of its homogeneous components is a 1-dimensional vector space over k_ν. If the residual extension is trivial, the same is true over k_R, and since the k_R-algebra $\mathrm{gr}_\nu R$ is a graded subalgebra of $\mathrm{gr}_\nu R_\nu$, each of its homogeneous components is a k_R-vector space of dimension ≤ 1. By an observation of Korkina ([3], see also [2]), this implies the result: taking a (possibly countable) system of homogeneous generators of the algebra gives a graded surjection $k_R[(U_i)_{i \in I}] \to \mathrm{gr}_\nu R$ once U_i is given the degree of its image. The kernel is generated by homogeneous polynomials, but any two terms of such a polynomial have non zero k_R-proportional images, which shows that the kernel is generated by binomials.

2) Let ν be a valuation of height one, i.e with archimedian value group $\Phi \subset \boldsymbol{R}$ (see [10], Vol. II). Assume first that Φ is generated by m rationally independent positive real numbers $\tau_1, \ldots, \tau_m$. We use the Perron algorithm as expounded in ([9], B. I, p. 861), but with a somewhat different interpretation. The algorithm consists in writing