A generating function for rational curves on rational surfaces

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1. Introduction. In [4], Yamagishi constructed families of elliptic curves of Mordell-Weil rank at least five and with a nontrivial rational 2-torsion point parametrized by certain rational curves. Her method is to construct a kind of universal family of such elliptic curves whose parameter space is a (2,2,2)-complete intersection in $\boldsymbol{P}^5$ and to find appropriate rational curves in it using the fact that it has a structure of elliptic surface. On the other hand, in [1], Bremner considered the problem to find all the rational curves of low degrees on a specific $K3$ surface. In view of their results, it should be of some interest to consider the following problem:

Given a projective surface S and a positive integer d, can one determine the number of algebraic equivalence classes of rational curves of degree d as explicitly as possible?

In order to detect the behavior of the number as a function of d, it will be useful to consider its generating function (see Section 2 for its definition). One of the main results of this paper is that for a class of important rational surfaces like Del Pezzo surfaces, rational ruled surfaces, the corresponding generating functions can be computed explicitly and that they are always *rational functions in x.*

Since for these surfaces we know the structure of their Néron-Severi groups quite explicitly, we can translate the problem into the one to solve a family of Diophantine equations parametrized by the degree of rational curves (see Section 3, for example, for the typical case of equation we must consider). Details will appear elsewhere.

2. Main theorem. Let k denote an arbitrary algebraically closed field. In this section we define a kind of counting function of rational curves on projective surfaces over k, and formulate our main theorem about the function.

Let S denote a nonsingular projective surface defined over k, and let $NS(S)$ denote its Néron-Severi group. We fix a projective imbedding of S, and denote by $H \in NS(S)$ the class of its hyperplane section. We consider the set R_d of algebraic equivalence classes of irreducible curves of arithmetic genus zero with degree d (w.r.t. H) on S. Let r_d denote the number of elements in R_d. (We will see below these numbers are finite.) The main concern of the present article is the following generating function of these numbers:

Definition 2.1. *We denote by $Z_{rat}(S\,;x)$ the formal power series $1 + \sum_{d\geq 1} r_d x^d$, and call it* zeta function for rational curves on S.

Theorem 2.2. *The zeta function for rational curves $Z_{rat}(\boldsymbol{P}^2\,;x)$ of $\boldsymbol{P}^2$ is given by*

$$Z_{rat}(\boldsymbol{P}^2\,;x) = 1 + \sum_{d\geq 1} x^d = \frac{1}{1-x}.$$

Let X_e denote the rational ruled surface $\boldsymbol{P}(O_{\mathrm{P}^1} \oplus O_{\mathrm{P}^1}(-e))$. We fix a section C_0 of X_e with $C_0^2 = -e$. Let f denote the algebraic equivalence class of its fiber. Then we know that $NS(X_e)$ is generated by C_0 and f (see [2, Chapter V], for example).

Theorem 2.3. *Let X_e denote the rational ruled surface $\boldsymbol{P}(O_{\mathrm{P}^1} \oplus O_{\mathrm{P}^1}(-e))$, regarded as embedded into a projective space by means of the very ample divisor $C_0 + (e+1)f$ in the notation of* [2]. *Then the zeta function for rational curves $Z_{rat}(X_e\,;x)$ is given by*

$$Z_{rat}(X_e\,;x) = \begin{cases} \dfrac{1+x-x^2+x^3}{1-x}, & \textit{if } e = 0, \\ \dfrac{1+x-x^2}{1-x}, & \textit{if } e = 1, \\ \dfrac{1+x-2x^2+x^{e+1}}{1-x}, & \textit{if } e \geq 2. \end{cases}$$

Remark. In general, the most general very ample divisor on X_e is given by $H_{m,n} = mC_0 + (e+n)f$, m, $n > 0$. If we regard X_e as embed-