Periodic Solutions of the Heat Convection Equations in Exterior Domains

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1. Introduction. Let $\Omega = K^c \subset \boldsymbol{R}^3$ where K is a compact set whose boundary ∂K is of class C^2. We put $\partial\Omega = \Gamma = \partial K$, $\hat{\Gamma} = \Gamma \times (0, \infty)$ and $\hat{\Omega} = \Omega \times (0, \infty)$. Then we consider the periodic problem for the heat convection equation (HCE):

$$(1)\quad \begin{cases} u_t + (u\cdot\nabla)u = -(\nabla p)/\rho + \{1 - \alpha(\theta - \Theta_0)\}g + \nu\Delta u & \text{in } \hat{\Omega}, \\ \operatorname{div} u = 0 & \text{in } \hat{\Omega}, \\ \theta_t + (u\cdot\nabla)\theta = \kappa\Delta\theta & \text{in } \hat{\Omega}, \end{cases}$$

$$(2)\ u(x, t)|_{\hat{\Gamma}} = 0,\ \theta(x, t)|_{\hat{\Gamma}} = \chi(x, t)\ (> 0), \quad \lim_{|x|\to\infty} u(x) = 0,\ \lim_{|x|\to\infty} \theta(x) = 0, \text{ for } t > 0,$$

$$(3)\quad u(\cdot, T) = u(\cdot, 0),\ \theta(\cdot, T) = \theta(\cdot, 0).$$

Here $u = u(x)$ is the velocity vector, $p = p(x)$ is the pressure and $\theta = \theta(x)$ is the temperature; ν, κ, α, ρ and $g = g(x)$ are the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion, the density at $\theta = \Theta_0$ and the gravitational vector, respectively. As for the exterior problem of (HCE), Hishida [2] showed the global existence of the strong solution for the initial value problem (IVP) in the case that K is a ball. Recently, Ōeda-Matsuda [7] showed the existence and uniqueness of weak solutions of (IVP) when K is a compact set with the boundary of class C^2. Moreover, Ōeda [10] obtained the stationary weak solutions for the similar exterior domain to that of [7]. In [7] and [10], we used "the extending domain method" to get weak solutions. Namely, it is expected that the exterior domain Ω can be approximated by interior domains $\Omega_n = B_n \cap \Omega$ (B_n is a ball with radius n and center at O) as $n \to \infty$ (see Ladyzhenskaya [3]). The purpose of the present paper is to show the existence of periodic weak solutions of (HCE) by using "the extending domain method".

2. Preliminaries. We make several assumptions: (A1) $\omega_0 \subset \operatorname{int} K$ (ω_0 being a neighbourhood of the origine O) and $K \subset B = B(O, d)$, where B is a ball with radius d and center at O. (A2) $\partial\Omega = \Gamma = \partial K \in C^2$. (A3) $g(x)$ is a bounded and continuous vector function in $\boldsymbol{R}^3 \setminus \omega_0$. Moreover there exist $R_0 > 0$, $C_{R_0} > 0$ such that $|g| \le C_{R_0}/|x|^{\frac{5}{2}+\varepsilon}$ for $|x| \ge R_0$ ($\varepsilon > 0$ is arbitrary). (A4) $\chi \in C^2(\Gamma \times [0, \infty))$ and is periodic with respect to t with period T.

Remark 1. Thanks to (A3), we see $g \in L^p(\Omega)$ for $p \ge \dfrac{6}{5}$.

We prepare a lemma which gives us an auxiliary function (see [1] p. 131 and [11] p.175):

Lemma 2.1. *There is a function* $\bar{\theta}(x, t)$ *which possesses the following properties* (i) ~ (iv): (i) $\bar{\theta} = \chi$ *on* $\hat{\Gamma}$. (ii) $\bar{\theta}(x, t) \in C_0^2(\boldsymbol{R}_x^3)$ *for any fixed* t *and* θ, θ_t *are continuous for* $t \in [0, T]$. (iii) $\bar{\theta}$ *is periodic in* t *with period* T. (iv) *For any* $\varepsilon > 0$ *and* $p > 1$, *we can retake* $\bar{\theta}$, *if necessary, such that* $\sup_{t\in[0,T]} \| \bar{\theta}(t) \|_{L^p} < \varepsilon$.

Now we make a change of variable: $\theta = \hat{\theta} + \bar{\theta}$, and after changing of variable, we use the same letter θ. Equations (1), (2), and (3) are transformed to the following:

$$(4)\quad \begin{cases} u_t + (u\cdot\nabla)u = -(\nabla p)/\rho - \alpha\theta g + \nu\Delta u + \{1 - \alpha(\bar{\theta} - \Theta_0)\}g & \text{in } \hat{\Omega}, \\ \operatorname{div} u = 0 & \text{in } \hat{\Omega}, \\ \theta_t + (u\cdot\nabla)\theta = \kappa\Delta\theta - (u\cdot\nabla)\bar{\theta} - \bar{\theta}_t + \kappa\Delta\bar{\theta} & \text{in } \hat{\Omega}, \end{cases}$$

$$(5)\quad u|_{\hat{\Gamma}} = 0,\ \theta|_{\hat{\Gamma}} = 0,\ \lim_{|x|\to\infty} u(x) = 0, \quad \lim_{|x|\to\infty} \theta(x) = 0,$$

$$(6)\quad u(\cdot, T) = u(\cdot, 0),\ \theta(\cdot, T) = \theta(\cdot, 0).$$

We put $G = \Omega$ or Ω_n, $\hat{G}' = G \times [0, T]$ and $\widehat{G \cup \Gamma}' = (G \cup \Gamma) \times [0, T]$. Then we write $W^{k,p}(G) = \{u; D^\alpha u \in L^p(G), |\alpha| \le k\}$, $W_0^{k,p}(G)$ = the completion of $C_0^k(G)$ in $W^{k,p}(G)$,
$D_\sigma(G) = \{\varphi \in C_0^\infty(G); \operatorname{div}\varphi = 0\}$, $D(G) = \{\phi \in C_0^\infty(G \cup \Gamma); \phi(\Gamma) = 0\}$,
$H_\sigma(G)$ (resp. $H_\sigma^1(G)$) = the completion of $D_\sigma(G)$ in $L^2(G)$ (resp. $W^{1,2}(G)$),
$H_0^1(\Omega_n)$ = the completion of $D(\Omega_n)$ in $W^{1,2}(\Omega_n)$ (it turns out $H_0^1(\Omega_n) = W_0^{1,2}(\Omega_n)$),
V(resp. W) = the completion of $D_\sigma(\Omega)$ (resp. $D(\Omega)$) in $\|\cdot\|_{N(\Omega)}$, where $\|u\|_{N(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$,
$\hat{D}_\sigma(\hat{G}) = \{\varphi \in C_0^\infty(\hat{G}'); \operatorname{div}\varphi = 0\}$, $\hat{D}(\hat{G}) = \{\phi$