Monopoles and Dipoles in Biharmonic Pseudo Process

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Δ^2 denotes the differential operator Laplacian square. It is called *the biharmonic operator* and plays an important role in the theory of elasticity and fluid dynamics. Given an equation

(1) $\partial_t u(t, x) = -\Delta^2 u(t, x) \equiv -\partial_x^4 u(t, x), \quad t > 0,\ x \in \mathbf{R}^1,$

we easily obtain its fundamental solution $p(t, x)$;

(2) $p(t, x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} d\xi \exp\{-i\xi x - \xi^4 t\}, \quad t > 0,\ x \in \mathbf{R}^1.$

Following the pioneer works of Krylov [2] and Hochberg [1], we consider 'particles' whose 'transition probability density' is taken to be $p(t, x)$ though $p(t, x)$ is not positive. We call these 'particles' *biharmonic pseudo process* (or *BPP* in short). In this note, we shall calculate a 'distribution' of the first hitting time of *BPP*, and it will be proved that *BPP* observed at a fixed point behaves as a mixture of particles of two different types, which are *'monopoles'* and *'dipoles'*.

1. Given positive t and s, $p(t, x)$ is an even function in x belonging to the Schwartz class $\mathcal{S}$, $p(t, x) = t^{-1/4} p(1, x/t^{1/4})$, $\int_{-\infty}^{\infty} dx\, p(t, x) = 1$, and $\int_{-\infty}^{\infty} dy\, p(t, y - x)\, p(s, y) = p(t + s, x)$. Here, note that values of $p(t, x)$ may be negative. In fact, Hochberg [1] proved that

$$p(1, |x|) = a|x|^{-1/3} \exp\{-b|x|^{4/3}\} \times \cos c|x|^{4/3} + \text{a lower order term}$$

for large $|x|$, where a, b, and c are positive constants. From the above, we see that

(3) $\displaystyle\int_{-\infty}^{\infty} |p(t, x)|dx = \int_{-\infty}^{\infty} |p(1, x)|\, dx = \text{a constant} \equiv \rho > 1.$

Basing on this $p(t, x)$, we can build up a finitely additive signed measure $\tilde{\boldsymbol{P}}_x$ on cylinder sets in $\mathbf{R}^{[0,\infty)}$. *A cylinder set* in $\mathbf{R}^{[0,\infty)}$, say Γ, is a set such that

(4) $\Gamma = \{\omega \in \mathbf{R}^{[0,\infty)} : \omega(t_1) \in B_1, \dots, \omega(t_n) \in B_n\}$

where $0 \le t_1 < \cdots < t_n$ and B_k's are Borel sets in $\mathbf{R}^1$. For each cylinder set as in (4), we define a finitely additive signed measure $\tilde{\boldsymbol{P}}_x$ by

(5) $\tilde{\boldsymbol{P}}_x[\Gamma] \equiv \displaystyle\int_{B_1} dy_1 \cdots \int_{B_n} dy_n\, p(t_1, y_1 - x) \times p(t_2 - t_1, y_1 - y_2) \cdots p(t_n - t_{n-1}, y_n - y_{n-1}).$

Fix $0 \le t_1 < \cdots < t_n$, and $\tilde{\boldsymbol{P}}_x$ is a σ-additive finite measure on a Borel field on $\mathbf{R}^n$ with total variation ρ^n. We say that a function f defined on $\mathbf{R}^{[0,\infty)}$ is *tame*, if

(6) $f(\omega) = g(\omega(t_1), \dots, \omega(t_n)),\ \omega \in \mathbf{R}^{[0,\infty)}$

for a Borel function g on $\mathbf{R}^n$ with $0 \le t_1 < \cdots < t_n$. For each tame function f as in (6), we define its expectation in the ordinary way;

(7) $\tilde{\boldsymbol{E}}_x[f(\omega)] = \displaystyle\int f(\omega)\, \tilde{\boldsymbol{P}}_x[d\omega(t_1) \times \cdots \times d\omega(t_n)]$

if the right hand side exists.

$\tilde{\boldsymbol{P}}_x$ satisfies the consistency condition, but (3) disturbs validity of Kolmogorov's extension theorem. So we do not know exactly an existence of a σ-additive extension of (5) into a function space. But we easily see that total variation of such σ-additive extension must be infinite if it may exist.

2. We extend an expectation given by (7). Let n and N be natural numbers. For each $\omega \in \mathbf{R}^{[0,\infty)}$, we set

$$\omega_n^N(t) \equiv \begin{cases} \omega\left(\dfrac{k}{2^n}\right) & \text{if } \dfrac{k}{2n} \le t < \dfrac{k+1}{2^n} \text{ and } t < N \\ \omega(N) & \text{if } t \ge N. \end{cases}$$

This approximating function ω_n^N is a step function in *Skorokhod space* $\mathbf{D}[0, \infty)$, which is a space of all right continuous functions on $[0, \infty)$ with left hand limits.

Definition 1. We say that a function F on $\mathbf{R}^{[0,\infty)}$ is *admissible* if F satisfies the following;

(a) for each n and N, $F(\omega_n^N)$ is tame,

(b) for each $\omega \in \mathbf{R}^{[0,\infty)}$, $\lim_{n\to\infty} \lim_{N\to\infty} F(\omega_n^N) = F(\omega)$,

(c) there exists $\lim_{n\to\infty} \lim_{N\to\infty} \sum_{k=1}^{N} |\tilde{\boldsymbol{E}}_x[F(\omega_n^k)] - \tilde{\boldsymbol{E}}_x[F(\omega_n^{k-1})]|$.

For each admissible function F, we define its expectation by

(8) $\boldsymbol{E}_x[F(\omega)] \equiv \lim\limits_{n\to\infty} \lim\limits_{N\to\infty} \tilde{\boldsymbol{E}}_x[F(\omega_n^N)].$

If exists, this expectation is unique owing to (c)