An Anticipatory Itô Formula

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1. Introduction. Let $B(t)$ be a Brownian motion. The well-known Itô formula states that for any C^2-function F on $\boldsymbol{R}$,

$$F(B(t)) = F(B(0)) + \int_0^t F'(B(s))dB(s) + \frac{1}{2}\int_0^t F''(B(s))ds,$$

where $\int_0^t F'(B(s))dB(s)$ is an Itô integral. Suppose θ is a C^2-function on $\boldsymbol{R}^2$. The purpose of this paper is to find an anticipatory Itô formula for $\theta(B(t), B(1))$. It is anticipatory because of the appearance of $B(1)$. In fact, we will give such a formula for $\theta(X(t), B(1))$ with $X(t)$ being a Wiener integral $X(t) = \int_0^t f(s)dB(s)$ such that $f \in L^\infty([0,1])$.

2. Hitsuda-Skorokhod integrals. Let $\mathscr{S}(\boldsymbol{R})$ denote the real Schwartz space on $\boldsymbol{R}$. The standard Gaussian measure on its dual space $\mathscr{S}'(\boldsymbol{R})$ is denoted by μ. Let (L^2) be the complex Hilbert space of square integrable functions on $(\mathscr{S}'(\boldsymbol{R}), \mu)$. Let $(\mathscr{S}) \subset (L^2) \subset (\mathscr{S})^*$ be a Gel'fand triple associated with $(\mathscr{S}'(\boldsymbol{R}), \mu)$ (see [2], [5], or [7]). Let ∂_t denote the white noise differentiation. It is a continuous linear operator from $(\mathscr{S})$ into itself. Its adjoint ∂_t^* is a continuous linear operator from $(\mathscr{S})^*$ into itself.

Let g be a weakly measurable function from [0,1] into $(\mathscr{S})^*$ such that $t \mapsto \partial_t^* g(t)$ is Pettis integrable. The integral $\int_0^1 \partial_t^* g(t)dt$ defines a generalized function in $(\mathscr{S})^*$. If it is a random variable in (L^2), then we call it the *Hitsuda-Skorokhod integral* of g ([3], [8]). For instance, if $g \in L^2([0,1] \times \mathscr{S}'(\boldsymbol{R}))$ is nonanticipating, then $\int_0^1 \partial_t^* g(t)dt$ is a Hitsuda-Skorokhod integral. In fact, for such a function g, its Hitsuda-Skorokhod integral agrees with its Itô integral [4], i.e.,

$$\int_0^1 \partial_t^* g(t)dt = \int_0^1 g(t)dB(t),$$

where $B(t)$ is the Brownian motion $B(t, x) = \langle x, 1_{[0,t)}\rangle$, $t \geq 0$, $x \in \mathscr{S}'(\boldsymbol{R})$. In particular, we have the equality

$$\left\| \int_0^1 \partial_t^* g(t)dt \right\|^2 = \int_0^1 \| g(t) \|^2 dt.$$

where $\| \cdot \|$ denotes the (L^2)-norm. However, this equality may hold even if g is not nonanticipating. For example, this equality holds for

$$g(t) = \begin{cases} B(t) + B(1) - B(1-t), & \text{if } 0 \leq t \leq \frac{1}{2}; \\ B(1-t) + B(1) - B(t), & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

3. An anticipatory Itô formula. Let $B(t)$ be the above Brownian motion. We have the following theorem.

Theorem 1. *Let $f \in L^\infty([0,1])$ and let $X(t) = \int_0^t f(s)dB(s)$, $t \in [0,1]$, be the Wiener integral of f. Suppose $\theta(x, y)$ is a C^2-function on $\boldsymbol{R}^2$ such that*

$$\theta(X(\cdot), B(1)), \frac{\partial^2\theta}{\partial x^2}(X(\cdot), B(1)),$$
$$\frac{\partial^2\theta}{\partial x\partial y}(X(\cdot), B(1)) \in L^2([0, 1] \times \mathscr{S}'(\boldsymbol{R})).$$

Then for any $0 \leq t \leq 1$, the integral $\int_0^t \partial_s^*(f(s)\frac{\partial\theta}{\partial x}(X(s), B(1)))ds$ is a Hitsuda-Skorokhod integral and the following equality holds in (L^2) for all $0 \leq t \leq 1$,

$$\begin{aligned}\theta(X(t), B(1)) &= \theta(X(0), B(1)) + \int_0^t \partial_s^*\Big(f(s)\frac{\partial\theta}{\partial x}(X(s), B(1))\Big)ds \\ &+ \int_0^t \Big(\frac{1}{2}f(s)^2\frac{\partial^2\theta}{\partial x^2}(X(s), B(1)) \\ &+ f(s)\frac{\partial^2\theta}{\partial x\partial y}(X(s), B(1))\Big)ds.\end{aligned}$$

To prove this theorem, we first assume that f is a simple function. In this case, we can use

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