

$$(4.3) \quad \begin{array}{ccc} \widetilde{Tr}_H(k) & \xrightarrow{\tilde{\tau}} & C_+(k) \\ & \searrow & \nearrow \\ & \tilde{g}_H(k) & \end{array}$$

where  $\tau : Tr_H(k) \rightarrow C_+(k)$  is given by  
 (4.4)  $\tau(t) = (\tan(A/2), \tan(B/2), \tan(C/2))$ .  
 All other notation in (4.3) should be self-explanatory and the proof goes similarly as before.

**Examples and comments.** When  $k = \mathbb{Q}$ , elements of  $Tr_H(\mathbb{Q})$  are called "rational triangles" or "Heron triangles" ([1] Chap. V). Heron of Alexandria noted that  $t = (13, 14, 15)$  belongs to  $Tr_H(\mathbb{Q})$  with  $\Delta = 84$ . By our map (4.4) it corresponds to the point  $(1/2, 4/7, 2/3)$  of the quadric  $C_+(\mathbb{Q})$ . On the other hand, by our map (1.6) it corresponds to the point  $(1/4, 16/49, 4/9)$  of the quartic  $S_+(\mathbb{Q})$ .

Obviously, every right triangle  $t = (a, b, c) \in Tr(k)$  belongs to  $Tr_H(k)$ . Assume that  $C = \pi/2$ ; hence  $a^2 + b^2 = c^2$ . Then  $\tau(t) = (a/(b+c), b/(a+c), 1)$  and  $\theta(t) = (a^2/(b+c)^2, b^2/(a+c)^2, 1)$ . In both cases the image of right triangles with  $C = \pi/2$  is the intersection of the surface in  $k_+^3$  and the plane  $z = 1$  (or  $w = 1$ ).

Needless to say, all equilateral triangles  $t = (a, a, a), a \in k_+$ , are similar and so they correspond to a single point in the quartic surface. If  $k$  does not contain  $3^{1/2}$ , then  $t \notin Tr_H(k)$  because  $\Delta_t = (3^{1/2}/4)a$ .

**§5. An involution.** For  $t = (a, b, c) \in Tr(k)$ ,

put  
 (5.1)  $t' = (a', b', c')$  with  $a' = a(s-a),$   
 $b' = b(s-b), c' = c(s-c), s = \frac{1}{2}(a+b+c).$

Then one finds  
 (5.2)  $s' - a' = (s-b)(s-c), s' - b' = (s-c)(s-a), s' - c' = (s-a)(s-b),$

with  $s' = \frac{1}{2}(a' + b' + c')$ . By (5.1), (5.2), we obtain a map:  $Tr(k) \rightarrow Tr(k)$ . Furthermore, for

the image  $t'' = (a'', b'', c'')$  of  $t' = (a', b', c')$ , we get

(5.3)  $a'' = a'(s' - a') = ad, b'' = bd, c'' = cd,$   
 with  $d = (s-a)(s-b)(s-c).$

In other words, we have  $t'' \sim t$  and so the map  $t \mapsto t'$  induces an involution  $*$  of  $\widetilde{Tr}(k)$ . The only fixed point of  $*$  is the class of equilateral triangle. By the diagram (2.5), we can transplant  $*$  on  $S_+(k)$  and  $\tilde{g}(k)$ . On the surface  $S_+(k)$ , the involution  $P = (x, y, z) \mapsto P = (x^*, y^*, z^*)$  is determined by the relation:

(5.4)  $xx^* = yy^* = zz^* = (xyz)/(xyz)^{1/2} + y(zx)^{1/2} + z(xy)^{1/2}.$

**Example** (Heron). Let  $k = \mathbb{Q}$  and  $t = (a, b, c) = (13, 14, 15) \in Tr(\mathbb{Q})$ . We have  $s = 21, s-a = 8, s-b = 7, s-c = 6, \Delta = (s(s-a)(s-b)(s-c))^{1/2} = 84$ , hence  $t \in Tr_H(\mathbb{Q})$ . Next, by (5.1), we have  $t' = (a', b', c') = (104, 98, 90), s' = 146$  and  $(\Delta')^2 = 16482816 = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$ , which means that  $t' \notin Tr_H(\mathbb{Q})$ ; in other words, the involution  $*$  of  $\widetilde{Tr}(\mathbb{Q})$  does not respect the subset  $\widetilde{Tr}_H(\mathbb{Q})$ . Passing to the surface  $S_+(\mathbb{Q})$ , we have

$\theta(t) = (1/2^2, 2^4/7^2, 2^2/3^2)$   
 $\theta(t)^* = (2^5/73, 7^3/(2 \cdot 73), (2 \cdot 3^2)/73).$

As for triples of elliptic curves, denoting by  $[P, Q]$  for the curve of type (3.1), we have

$E_t = (E_a, E_b, E_c) = ([126, -84^2], [99, \cdot], [70, \cdot]),$   
 $E_t^* = (E_{a'}, E_{b'}, E_{c'}) = ([3444, -2^9 \cdot 3^2 \cdot 7^2 \cdot 73], [4656, \cdot], [6160, \cdot]).$

**References**

[1] Dickson, L. E.: History of the Theory of Numbers. vol. 2, Chelsea, New York (1971).  
 [2] Ono, T.: Triangles and elliptic curves. I ~ VI. Proc. Japan Acad., **70A**, 106-108(1994); **70A**, 223-225 (1994); **70A**, 311-314 (1994); **71A**, 104-106 (1995); **71A**, 137-139 (1995); **71A**, 184-186 (1995).

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I take this opportunity to make a correction to my paper (VI). On p. 186, in (4.6),  $x^3 + 4x^2 - 3x$  should read  $x^3 + 2x^2 - 3x$ .