On Bloch-to-Besov Composition Operators

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1. Introduction. Let $H(D)$ be the space of analytic functions on the unit disk D. Every holomorphic self-map $\varphi : D \to D$ induces a linear composition operator C_φ from $H(D)$ into itself as follows: $C_\varphi f = f \circ \varphi$, whenever $f \in H(D)$. In this paper we consider composition operators from the Bloch space $\mathcal{B}$ to the spaces of analytic Besov functions B_p, $1 < p < \infty$.

Recall the definitions of the Bloch space $\mathcal{B}$ and the analytic Besov spaces B_p (see e.g. [9]).

The function f is called a Bloch function if it is analytic in D and

$$\| f \|_{\mathcal{B}} = \sup_{z \in D} (1 - |z|^2) |f'(z)| < \infty.$$

This defines a semi-norm. The Bloch functions form a Banach space $\mathcal{B}$ with the norm $\| f \| = |f(0)| + \| f \|_{\mathcal{B}}$.

The analytic Besov functions are defined as follows

$$B_p = \left\{ f \in H(D) : \| f \|_{B_p} = \left(\iint_D ((1 - |z|^2) |f'(z)|)^p d\lambda(z) \right)^{\frac{1}{p}} < \infty \right\},$$

where $d\lambda(z) = \dfrac{dA(z)}{(1 - |z|^2)^2}$ is the hyperbolic area measure on D and $dA(z) = \dfrac{1}{\pi} dx\, dy$.

The analytic Besov functions form a Banach space B_p, $1 < p < \infty$, with the norm $\| f \| = |f(0)| + \| f \|_{B_p}$.

Let B be the family of holomorphic self-maps φ of the unit disk D into itself. By the Schwarz-Pick lemma $\sup_{z \in D} (1 - |z|^2) \varphi^*(z) \leq 1$ for any $\varphi \in B$, where $\varphi^*(z)$ is the hyperbolic derivative

$$\varphi^*(z) = \frac{|\varphi'(z)|}{1 - |\varphi(z)|^2}.$$

We say that $\varphi \in B_0$ if

$$\lim_{|z| \to 1} (1 - |z|^2) \varphi^*(z) = 0.$$

Definition. *For $1 < p < \infty$ the hyperbolic analytic Besov class B_p^h is defined to be the family of all functions $\varphi \in B$ such that*

$$\| \varphi \|_{B_p^h} = \left(\iint_D ((1 - |z|^2) \varphi^*(z))^p d\lambda(z) \right)^{\frac{1}{p}} < \infty.$$

We can assume that $B_\infty^h = B$. However Möbius transforms of D are not p-hyperbolic Besov functions for $1 < p < \infty$.

Let $T_a(z) = \dfrac{a - z}{1 - \bar{a}z}$, $a \in D$, and $\varphi_a(z) = \varphi(T_a(z))$.

For every $\varphi \in B_p^h$ and every $a \in D$ functions $T_a \circ \varphi(z)$ and $\varphi \circ T_a(z)$ belong to the class B_p^h.

We give some examples of functions which are in B_p^h or are not in B_p^h.

1. Let $S_\alpha = \{ z = x + iy : |x|^\alpha + |y|^\alpha < 1 \}$, $0 < \alpha \leq 1$, and φ_α be a conformal mapping of D into S_α, then $\varphi_1 \notin B_2^h$. If $\alpha < 1$ then $\varphi_\alpha(z) \in B_2^h$.

2. Let φ be a bounded holomorphic function in D with $\| \varphi \|_\infty \leq k < 1$. If φ is continuous in $\bar{D}$ and $\varphi(e^{i\theta}) \in \Lambda_\alpha$, $0 < \alpha \leq 1$, then

$(1 - |z|^2) |\varphi'(z)| = O((1 - |z|^2)^\alpha)$ as $|z| \to 1$,

and also

$(1 - |z|^2) \varphi^*(z) = O((1 - |z|^2)^\alpha)$ as $|z| \to 1$

by the Hardy-Littlewood theorem ([3], Theorem 5.1). Thus $\varphi \in B_p^h$ for $p > \dfrac{1}{\alpha}$. See also [7].

2. Composition operators on the Bloch space. Our main results are the following

Theorem 1. *Let φ be a holomorphic mapping of D into itself and $1 < p < \infty$. C_φ is a Bloch-to-B_p composition operator if and only if $\varphi \in B_p^h$.*

Theorem 2. *If $\varphi \in B_p^h$, $1 < p < \infty$, then φ induces a compact composition operator C_φ on $\mathcal{B}$ into B_p.*

Proof of Theorem 1. Let φ be any function of B_p^h, $1 < p < \infty$, and f be any Bloch function. Then we obtain

$$\| f \circ \varphi \|_{B_p}^p = \iint_D (1 - |z|^2)^p |g'(z)|^p d\lambda(z)$$

$$= \iint_D (1 - |z|^2)^p |f' \circ \varphi(z)|^p |\varphi'(z)|^p d\lambda(z)$$