# A New Version of the Factorization of a Differential Equation of the Form $F(x,y,\tau y)=0$

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In this note, we will consider equations of the form

$$(\mathrm{E}_0)\qquad F(x, y, \tau y) = 0,$$

where $F(x, y, X)$ is a holomorphic function defined in a neighborhood of the origin of $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y \times \boldsymbol{C}_X$, and $\tau$ is a vector field

$$\tau = \textstyle\sum_{1 \leqslant i \leqslant n} \alpha_i(x, y)\partial/\partial x_i$$

with coefficients $\alpha_i(x, y)\,(1 \leqslant i \leqslant n)$ meromorphic in $x$ at most with only poles along a union of a finite number of hyperplanes (in $(\boldsymbol{C}_x)^n$) and holomorphic in $y$ near the origin of $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y$.

If $F(x, y, X)$ is of finite order, say $m$, with respect to the variable $X$ by Weierstrass preparation theorem $F(x, y, X) = 0$ is equivalent to

$$X^m + \textstyle\sum_{1 \leqslant j \leqslant m} a_j(x, y)X^{m-j} = 0$$

and $(\mathrm{E}_0)$ is reduced to

$$(\mathrm{E})\qquad (\tau y)^m + \textstyle\sum_{1 \leqslant j \leqslant m} a_j(x, y)(\tau y)^{m-j} = 0.$$

In our previous paper [1] we have presented a factorization theorem for (E) which asserts that (E) is factorized into a product of equations of the form $\tau y = f(x, y)$ near the point $x = 0$. In this note we will present a new version of this theorem.

**§1. Factorization theorems.** Let us consider the following differential equation:

$$(\mathrm{E})\qquad F(x, y, \tau y) = (\tau y)^m + \textstyle\sum_{1 \leqslant j \leqslant m} a_j(x, y)(\tau y)^{m-j} = 0$$

where $m \in \boldsymbol{N}^*(= \{1, 2, \ldots\})$, $x = (x_1, \ldots, x_n) \in \boldsymbol{C}^n$, $n \in \boldsymbol{N}^*$, $y \in \boldsymbol{C}$, and $a_j(x, y)\,(1 \leqslant j \leqslant m)$ are holomorphic functions defined in a neighborhood of the origin (0,0) of $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y$. In (E), $y = y(x)$ is regarded as an unknown function of $x$ and $\tau$ is a vector field of the form

$$\tau = \textstyle\sum_{1 \leqslant i \leqslant n} \alpha_i(x, y)\partial/\partial x_i$$

whose coefficients $\alpha_i(x, y)\,(1 \leqslant i \leqslant n)$ are meromorphic in $x$ at most with only poles along a union of a finite number of hyperplanes (in $(\boldsymbol{C}_x)^n$) and holomorphic in $y$ in a neighborhood of the origin $(x, y) = (0,0)$ in $(\boldsymbol{C}_x)^n \times \boldsymbol{C}_y$.

**Definition 1.** We say that the transformation

$$x = (x_1, \ldots, x_n) \rightarrow t = (t_1, \ldots, t_n)$$

is of type $(GT)$ if it is defined by the following: first we transform $x = (x_1, \ldots, x_n) \rightarrow \xi = (\xi_1, \ldots, \xi_n)$ by $x = A\xi$ for some $A \in GL(n, \boldsymbol{C})$ and then we transform $\xi \rightarrow t$ by

$$\xi_1 = (t_1)^k,\ \xi_2 = (t_1)^k t_2, \ldots,\ \xi_n = (t_1)^k t_n$$

for some $k \in \boldsymbol{N}^*$.

The result of our previous paper [1] is as follows:

**Theorem 1 ([Theorem 2.2; 1]).** *After a suitable transformation $x \rightarrow t$ which is obtained by a composition of a finite number of transformations of type $(GT)$, we can choose $c \in \boldsymbol{C}$ such that the following conditions hold:*

1) *$c = 0$ or $|c|$ is sufficiently small;*

2) *by setting $y = c + z$ the equation* (E) *is decomposed in a neighborhood of the origin* $(0,0) \in (\boldsymbol{C}_t)^n \times \boldsymbol{C}_z$ *into the form*

$$(1.1)\qquad \Pi_{1 \leqslant j \leqslant m}(\tau^* z - \varphi_j(t, z)) = 0,$$

*where $\tau^*$ is the transform of $\tau$ by the transformation $x \rightarrow t$ and $\varphi_j(t, z)\,(1 \leqslant j \leqslant m)$ are holomorphic functions defined in a neighborhood of* $(0,0) \in (\boldsymbol{C}_t)^n \times \boldsymbol{C}_z$.

Note that the original equation (E) is considered near $(x, y) = (0,0)$; but the decomposition (1.1) is obtained in a neighborhood of $(x, y) = (0, c)$ which may exclude the point $(x, y) = (0,0)$ in case $c \neq 0$. Therefore, if we want to study the behaviour of the solutions of (E) near the origin (0,0) we must fill some gaps between (E) and (1.1).

To fill up the gap we will present here a new version of factorization theorem. In our new result, instead of using transformations of type $(GT)$ and a shift $y = c + z$ we will use the following transformation:

**Definition 2.** We say that the transformation

$$(x, y) = (x_1, \ldots, x_n, y) \rightarrow (t, z) = (t_1, \ldots, t_n, z)$$

is of type $(NGT)$ if it is defined by the follow-

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