

Einstein Normal Homogeneous Riemannian Manifold

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In this paper, we get a necessary and sufficient condition for certain normal homogeneous Riemannian manifolds with two irreducible summands by the isotropic representation to be Einstein. And then, we give such an example.

Let G be a compact connected semi-simple Lie group and H a closed subgroup. We denote by \mathfrak{g} and \mathfrak{h} the corresponding Lie algebras of G and H . Let B be the Killing form of \mathfrak{g} . Let g_0 be the normal homogeneous metric in G/H which is induced from $Q(:= -B)$. We consider the $Ad(H)$ -invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ with $Q(\mathfrak{h}, \mathfrak{m}) = 0$. Let $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ be a Q -orthogonal $Ad(H)$ -invariant decomposition such that $Ad(H)|_{\mathfrak{m}_i}$ is irreducible for $i = 1, 2$ and assume that \mathfrak{m}_1 and \mathfrak{m}_2 are inequivalent irreducible $Ad(H)$ -representation spaces such that
 (1) $[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}$ and $[\mathfrak{m}_2, \mathfrak{m}_2] \subset (\mathfrak{h} + \mathfrak{m}_1) =: \mathfrak{k}$.
 Let K be a closed connected subgroup with $H \subseteq K \subseteq G$ which has the subalgebra \mathfrak{k} as its Lie algebra.

In this paper, we assume $(G/H, g_0)$ is a compact normal homogeneous Riemannian manifolds satisfying condition (1). The space of G -invariant symmetric covariant 2-tensors on G/H is given by $\{x_1 Q|_{\mathfrak{m}_1} + x_2 Q|_{\mathfrak{m}_2} \mid x_1, x_2 \in R\}$. The Ricci tensor ρ of G -invariant Riemannian metric on G/H is a G -invariant symmetric covariant 2-tensor on G/H , and we identify ρ with an $Ad(H)$ -invariant symmetric bilinear form on \mathfrak{m} . Thus ρ is written as $\rho = r_1 Q|_{\mathfrak{m}_1} + r_2 Q|_{\mathfrak{m}_2}$ for some $r_1, r_2 \in R$.

Now, we compute components of Ricci tensor ρ of $(G/H, g_0)$ explicitly. Let $d_i = \dim_{\mathbb{R}} \mathfrak{m}_i (i = 1, 2)$. Let $\{e_\alpha\}$ be a Q -orthogonal basis adapted to the decomposition of \mathfrak{m} , i.e., each $e_\alpha \in \mathfrak{m}_i$ for some $i \in \{1, 2\}$, and $\alpha < \beta$ if $e_\alpha \in \mathfrak{m}_1$ and $e_\beta \in \mathfrak{m}_2$. Next set $A_{\alpha\beta}^r = Q([e_\alpha, e_\beta], e_r)$, so that $[e_\alpha, e_\beta]_{\mathfrak{m}} = \sum_r A_{\alpha\beta}^r e_r$, and set $\begin{bmatrix} k \\ ij \end{bmatrix} := \sum (A_{\alpha\beta}^r)^2$, where the sum is taken all over indices α, β, γ , with $e_\alpha \in \mathfrak{m}_i, e_\beta \in \mathfrak{m}_j, e_r \in \mathfrak{m}_k (1 \leq i, j, k \leq 2)$, and

$\begin{bmatrix} k \\ ij \end{bmatrix}_{\mathfrak{m}}$ denotes the \mathfrak{m} -component. Then, $\begin{bmatrix} k \\ ij \end{bmatrix}$ is independent of the Q -orthonormal bases chosen for $\mathfrak{m}_1, \mathfrak{m}_2$, and

$$(2) \quad \begin{bmatrix} k \\ ij \end{bmatrix} = \begin{bmatrix} k \\ ji \end{bmatrix} = \begin{bmatrix} j \\ ki \end{bmatrix}.$$

Lemma 1. *The components r_1, r_2 of Ricci tensor ρ on $(G/H, g_0)$ are given*

$$(3) \quad r_k = \frac{1}{2} - \frac{1}{4d_k} \sum_{j,i} \begin{bmatrix} k \\ ji \end{bmatrix} \quad (k = 1, 2).$$

Proof. Let $\{e_j^{(k)}\}_{j=1}^{d_k}$ be Q -orthonormal basis on $\mathfrak{m}_k (k = 1, 2)$. The Ricci tensor ρ on $(G/H, g_0)$ is given by the following (cf. [1], pp. 184–185):

$$\begin{aligned} \rho(X, X) &= -\frac{1}{2} \sum_{\alpha} Q([X, e_{\alpha}]_{\mathfrak{m}}, [X, e_{\alpha}]_{\mathfrak{m}}) \\ &\quad + \frac{1}{2} Q(X, X) + \frac{1}{4} \sum_{\beta, \alpha} Q([e_{\beta}, e_{\alpha}]_{\mathfrak{m}}, X)^2 \end{aligned}$$

for $X \in \mathfrak{m}$. From this equation, we have

$$\begin{aligned} r_k &= r(e_l^{(k)}, e_l^{(k)}) \\ &= \frac{1}{2} - \frac{1}{2} \sum_{j,i} \sum_s Q([e_l^{(k)}, e_s^{(i)}]_{\mathfrak{m}_j}, [e_l^{(k)}, e_s^{(i)}]_{\mathfrak{m}_j}) \\ &\quad + \frac{1}{4} \sum_{j,i} \sum_{s,t} Q([e_s^{(j)}, e_t^{(i)}]_{\mathfrak{m}_k}, e_l^{(k)})^2. \end{aligned}$$

As we remarked above,

$$d_k r_k = \sum_{\ell=1}^{d_k} r(e_{\ell}^{(k)}, e_{\ell}^{(k)}) = \frac{d_k}{2} - \frac{1}{4} \sum_{j,i} \begin{bmatrix} j \\ ki \end{bmatrix}.$$

Q.E.D.

Homogeneous space K/H in $H \subseteq K \subseteq G$ need not be effective in general. So let K' be the quotient of K acting effectively on K/H . We also assume that K' is semi-simple and $cQ|_{\mathfrak{k}'} = Q_{\mathfrak{k}'}$ for some $c > 0$, where $Q_{\mathfrak{k}'}$ is the negative of the Killing form of \mathfrak{k}' .

By our assumption (1), we have

$$(4) \quad \begin{bmatrix} 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ 22 \end{bmatrix} = 0.$$

We obtain

$$(5) \quad \begin{bmatrix} 2 \\ 12 \end{bmatrix} = (1 - c)d_1,$$

since