A Trace Formula for the Picard Group. I

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1. Statement of the result. The aim of the present note is to report the analogue of the Kuznetsov trace formula for the Picard group $\Gamma = PSL(2, \boldsymbol{Z}[i])$ acting discontinuously over the Beltrami model of the three dimensional Lobachevsky geometry (i.e., the hyperbolic upper half-space $\boldsymbol{H}^3$). Our argument is an extension of one of our proofs [3, Section 2.7] of the ordinary Kuznetsov trace formula. We stress that the Picard group has been chosen as a model case. In fact the argument can well be applied to Bianchi groups over arbitrary imaginary quadratic number fields with some extra technical complexities. Our trace formula should have applications to the analytic theory of imaginary quadratic number fields in much the same way as the original Kuznetsov formula has been applied to various important problems in the rational number field. The binary additive divisor problem over imaginary quadratic number fields is one of our targets. It is of particular importance because of its relation with the mean-value problem of Dedekind zeta-functions of respective fields. Such an application will, however, require an enhancement of our formula with the incorporation of Grössencharakters. To these topics and the details of the proof we shall return elsewhere.

To state our trace formula we need some definitions: We denote a point of $\boldsymbol{H}^3$ by $z = (x, y)$ with $x = x_1 + x_2 i$ ($x_1, x_2 \in \boldsymbol{R}$) and $y > 0$. Then the hyperbolic volume element is $d\mu(z) = y^{-3}dx_1dx_2dy$, and the hyperbolic Laplace-Beltrami operator is $\Delta = -y^2((\partial/\partial x_1)^2 + (\partial/\partial x_2)^2 + (\partial/\partial y)^2) + y(\partial/\partial y)$. The set of all Γ-invariant functions over $\boldsymbol{H}^3$ which are square integrable with respect to $d\mu$ over the hyperbolic three-manifold $\mathcal{T} = \Gamma \backslash \boldsymbol{H}^3$ constitutes the Hilbert space $L^2(\mathcal{T}, d\mu)$. The non-trivial discrete spectrum of Δ over $L^2(\mathcal{T}, d\mu)$ is denoted by $\{\lambda_j = 1 + \kappa_j^2 : j = 1,2, \dots\}$ where $\kappa_j > 0$, and the corresponding orthonormal system of eigenfunctions by $\{\psi_j\}$. We have the Fourier expansion

$$\psi_j(z) = y \sum_{n \in \boldsymbol{Z}[i], n \neq 0} \rho_j(n) K_{i\kappa_j}(2\pi |n| y) e(\langle n, x \rangle).$$

Here K_ν is the K-Bessel function of order ν, $e(a) = \exp(2\pi i a)$, and $\langle n, x \rangle = \mathrm{Re}(n\bar{x})$. We introduce also the Kloosterman sum

$$S(m, n; l) = \sum_{v (\mathrm{mod}\ l), (v, l) = 1} e(\langle m, v/l \rangle + \langle n, v^*/l \rangle), \quad (l, m, n \in \boldsymbol{Z}[i]),$$

where $vv^* \equiv 1 \bmod l$. Further we shall need the Dedekind zeta-function ζ_K of $\boldsymbol{Q}(i)$ as well as the divisor function $\sigma_\nu(n) = \frac{1}{4}\sum_{d|n} |d|^{2\nu}$ $(n, d \in \boldsymbol{Z}[i])$.

Our trace formula is embodied in

Theorem. *Let us assume that the function $h(r)$, $r \in \boldsymbol{C}$, is regular in the horizontal strip $|\mathrm{Im}\, r| < \frac{1}{2} + \varepsilon$ and satisfies*

$$h(r) = h(-r), \quad h(r) \ll (1 + |r|)^{-3-\varepsilon}$$

with an arbitrary fixed $\varepsilon > 0$. Then we have, for any non-zero $m, n \in \boldsymbol{Z}[i]$,

$$\sum_{j=1}^{\infty} \frac{\overline{\rho_j(m)\rho_j(n)}}{\sinh(\pi\kappa_j)} \kappa_j h(\kappa_j) + 2\pi \int_{-\infty}^{\infty} \frac{\sigma_{ir}(m)\sigma_{ir}(n)}{|mn|^{ir} |\zeta_K(1 + ir)|^2} h(r)\, dr$$
$$= (\delta_{m,n} + \delta_{m,-n})\pi^{-2} \int_{-\infty}^{\infty} r^2 h(r)\, dr + \sum_{l \in \boldsymbol{Z}[i], l \neq 0} |l|^{-2} S(m, n; l) \check{h}(2\pi\varpi)$$

with $\varpi^2 = \overline{mn}/l^2$. Here $\delta_{m,n}$ is the Kronecker delta, and

$$\check{h}(t) = i \int_{-\infty}^{\infty} \frac{r^2}{\sinh(\pi r)} J_{ir}(t) J_{ir}(\bar{t}) h(r)\, dr$$

with J_ν being the J-Bessel function of order ν.

2. Sketch of the proof. First we introduce the non-holomorphic Poincaré series over Γ: For $m \in \boldsymbol{Z}[i]$ we put

$$P_m(z, s) = \sum_{\gamma \in \Gamma_t \backslash \Gamma} y(\gamma z)^s \exp(-2\pi |m| y(\gamma z) + 2\pi i \langle m, x(\gamma z) \rangle) \quad (z \in \boldsymbol{H}^3, s \in \boldsymbol{C}),$$

where Γ_t is the translation subgroup in Γ (see Sarnak [5]). Expanding this into a double Fourier series with respect to the variables x_1, x_2 we get immediately