Inverse Mapping Theorem in the Ultradifferentiable Class

By Minoru KOIKE

Department of Mathematics Shibaura Institute of Technology

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The main purpose of this paper is to give a simple proof of a result similar to the inverse mapping theorem of Komatsu [1] under a weaker condition than that of [1], including the infinite dimensional case treated in Yamanaka [3].

In [1],[3] the majorant series method and the Lagrange formula are used, and [3] uses a generalization of the higher order chain rule of Faa'di Bruno. Here neither the majorant series method nor the higher order chain rule is utilized. Alternatively we prove and use a generalization of a result in Rudin [2] and a variant of the Lagrange formula (Theorem 3 below).

Let M_p, $p = 0,1,2,\cdots$, be a sequence of positive numbers with $M_1 = 1$. Let X, Y be Banach spaces and U an open subset of X. A map $f : U \to Y$ is said to belong to the ultradifferentiable class $\{M_p\}$ (or $\{M_p\}(U, Y)$), if $f \in C^\infty(U, Y)$ in the sense of Fréchet-differentiation and if there are constants C and h such that

$$\| f^{(p)}(x) \| \le Ch^p M_p, \quad p = 0,1,2,\cdots, x \in U.$$

In [2], [3] the following condition is considered: There is a constant H such that

$$(1) \qquad N_p^{1/p} \le HN_q^{1/q} \quad \text{if } 1 \le p \le q,$$

where

$$N_p = \frac{M_p}{p!}.$$

Here we consider the condition that there is a constant H such that the inequality

$$(2) \qquad \prod_{i=1}^{j} N_{k_i} \le H^n N_n$$

holds for positive integers k_i with $\sum_{i=1}^{j} k_i = n$, $n = 1,2,\cdots$, $j = 1,2,\cdots, n$.

This condition follows from (1).

Example. For $n = 1,2,\cdots$, let

$$M_n = \begin{cases} n!n^{n(n-1)} & (n = 2^m, m = 0,1,\cdots) \\ n!n^{n(n+1)} & \text{(otherwise)}. \end{cases}$$

Then this sequence $\{M_p\}$ satisfies (2) with $H = 1$ but not the condition (1). In fact we have $\sup\{N_{n-1}^{1/(n-1)}/N_n^{1/n}; n = 2^m, m \ge 1\} = \infty$. On the other hand, if $\sum_{i=1}^{j} k_i = n$ and $1 \le k_i < n - 1$, then

$$\prod_{i=1}^{j} N_{k_i} \le \prod_{i=1}^{j} k_i^{k_i(k_i+1)} \le \prod_{i=1}^{j} n^{k_i(n-1)} \le N_n.$$

If $k_r = n - 1$ for some r, then $j = 2$ and $k_s = 1$ $(s \ne r)$, hence $\prod_{i=1}^{j} N_{k_i} = N_{n-1} \le N_n$.

Thus (2) is strictly weaker than (1).

It is shown in [2] that the class $\{M_p\}$ is closed under division (in the one-dimensional case) if M_p satisfies (1). Here we have the following generalization of this.

Theorem 1. Assume (2). Let X, Y and Z be Banach spaces and U an open subset of X. If T belongs to the class $\{M_p\}(U, L(Z, Y))$ and $T(a) : Z \to Y$ is bijective for a point a in U, then the map $x \mapsto [T(x)]^{-1}$ belongs to the class $\{M_p\}(U_0, L(Y, Z))$ for some open subset U_0 of U containing a.

Proof. By assumption we have

$$\| T^{(k)}(x) \| \le h^{k+1} M_k, \quad k = 0,1,2,\cdots,$$

with some constant h. The open mapping theorem implies that $[T(a)]^{-1}$ belongs to $L(Y, Z)$. There exists an open set U_0 containing a such that, for $x \in U_0$, $[T(x)]^{-1}$ coincides with

$$R(x) = [T(a)]^{-1} \sum_{j=0}^{\infty} \{(T(a) - T(x))[T(a)]^{-1}\}^j,$$

which belongs to $L(Y, Z)$ and $\| R(x) \| \le C$ for a constant C. By the boundedness of derivatives of T and by the Leibniz rule, the series

$$R(u) = R(x) \sum_{j=0}^{\infty} [(T(x) - T(u))R(x)]^j$$

may be differentiated with respect to u in a neighborhood of x, term by term any number of times, since the resulting series converge uniformly in the neighborhood of x. Putting $u = x$ after differentiating this equality n-times by u, we have

$$R^{(n)}(x) = R(x) \sum_{j=1}^{n} \sum n! \prod_{i=1}^{j} \frac{1}{k_i!} [- T^{(k_i)}(x)R(x)],$$

where $\sum$ denotes the summation with respect to positive integers k_i with $\sum_{i=1}^{j} k_i = n$. Thus (2) implies

$$\| R^{(n)}(x) \| \le C \sum_{j=1}^{n} \sum n! \prod_{i=1}^{j} Ch^{k_i+1} \frac{M_{k_i}}{k_i!}$$