Criterion of Wiener Type for Minimal Thinness on Covering Surfaces

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Introduction. M. Lelong [6] and L. Naïm [8] obtained a criterion of Wiener type for minimal thinness for the Martin compactification of the upper half space of the d-dimensional Euclidean space $(d > 1)$. The purpose of this note is to give a criterion of Wiener type for minimal thinness for the Martin compactification of a finite sheeted covering surface of a punctured Riemannian sphere. It is sufficient to consider an r-sheeted unlimited covering surface W of $D - \{0\}$ (D is the unit disc). Denote by ∂W the relative boundary of W and $\pi = \pi_W$ the projection of $\bar{W} = W \cup \partial W$ onto $\{0 < |z| \leq 1\}$. We consider the Martin compactification W^* of W. Then W^* takes a form $W^* = W \cup \partial W \cup \Delta$, where Δ is the *ideal boundary* of a bordered surface $\bar{W}$. We also denote by Δ_1 the set of minimal points in Δ. We note that $1 \leq \# \Delta_1 \leq r$, where $\# \Delta_1$ is the number of points in Δ_1 (cf. [4]). Let $\Delta_1 = \{\zeta_1, \ldots, \zeta_m\}$ $(m = \# \Delta_1)$ and denote by $k_j = k_{\zeta_j}$ $(j = 1, \ldots, m)$ the Martin function with pole at ζ_j. We set $U_j = \{w \in W : k_j(w) > \sum_{i \neq j} k_i(w)\}$ $(j = 1, \ldots, m)$ in the case of $m > 1$ and $U_1 = W$ in the case of $m = 1$.

Main theorem. *Let E be a subset of W and j be an integer with $1 \leq j \leq m$. Set $E_n = \{w \in E \cap U_j : s^n \leq k_j(w) \leq s^{n+1}\}$ $(s > 1)$. Then, E is minimally thin at ζ_j if and only if*

$$\sum_{n=1}^{\infty} cap_W(E_n) s^n < +\infty,$$

where $cap_W(E_n)$ is the outer Green capacity of E_n.

1. Preliminaries 1.1 We begin with recalling the definition of balayage. Consider an open Riemann surface F possessing the Green function. Denote by $\mathcal{S} = \mathcal{S}(F)$ the class of all nonnegative superharmonic functions on F. Let E be a subset of F and s belong to $\mathcal{S}$. Then the *balayage* $\hat{R}_s^E = {}^F\hat{R}_s^E$ of s relative to E on F is defined by

$$\hat{R}_s^E(z) = \liminf_{x \to z} \inf\{u(x) : u \in \mathcal{S},\ u \geq s \text{ on } E\}$$

(cf. e.g. [2]). For informations about fundamental properties of balayage we refer to [1],[2], [5], etc.

The following lemma gives us the relation between the balayage on F and that on a covering surface of F.

Lemma 1.1 (cf. [7]**).** *Let $\tilde{F}$ be an unlimited covering surface of F, E a subset of F, s a positive superharmonic function on F and π the canonical projection from $\tilde{F}$ onto F. Then, it holds that*

$$^F\hat{R}_s^E \circ \pi = {}^{\tilde{F}}\hat{R}_{s\circ\pi}^{\pi^{-1}(E)}$$

on $\tilde{F}$.

Next we state the definition of thinness (cf. [1]). Let G_z^F be the Green function on F with pole at z.

Definition 1.1. *Let z be a point of F and E a subset of F. We say that E is thin at z if ${}^F\hat{R}_{G_z^F}^E \neq G_z^F$ on F.*

Assuming that E is closed and z belongs to E in the above definition, it is well-known that E is thin at z if and only if z is an irregular point of $F - E$ with respect to Dirichlet problem (cf. e.g. [2]). In the case of $F = D = \{z \in \mathbf{C} : |z| < 1\}$ we here review the Wiener criterion for thinness.

Proposition 1.1 (cf. [1]). *Let L be a subset of D. Set*

$$L_n = \{z \in L : s^n \leq \log|z|^{-1} \leq s^{n+1}\}\ (s > 1).$$

Then, L is thin at 0 if and only if

$$\sum_{n=1}^{\infty} cap_D(L_n) s^n < +\infty,$$

where $cap_D(L_n)$ is the outer Green capacity of L_n.

1.2. First we begin with definition of minimal thinness. Let k_ζ be the Martin function on F with pole at $\zeta \in \Delta_1^F$.

Definition 1.2 (cf. [1]). Let ζ be a point of Δ_1^F and E a subset of F. Then, we say that E is *minimally thin* at ζ if $\hat{R}_{k_\zeta}^E \neq k_\zeta$ on F.

Definition 1.3. Let ζ be a point of Δ_1^F and U a subset of F. We say that $U \cup \{\zeta\}$ is *a minimal fine neighborhood* of ζ if $F - U$ is minimally thin at ζ.