Oscillation of Solutions of Nonlinear Wave Equations*)

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1. Introduction. In this paper we shall consider the sign of solutions of certain nonlinear wave equations subject to suitable homogeneous boundary conditions. It is related to the oscillation behavior of continuous finite bodies with respect to the time variable.

Let Ω be a bounded simply connected domain in $\boldsymbol{R}^n$ and $\partial\Omega$ be its smooth boundary.

We suppose all functions and solutions appeared in this paper to be real-valued. We denote $\frac{\partial}{\partial x_k}$ $(k = 1,2\dots n)$ by ∂_k and $\frac{\partial}{\partial t}$ by ∂_t.

We shall consider the nonlinear wave equation

(1) $$\mathscr{D}u = \partial_t(\alpha(t)\partial_t u) + \beta(t)\partial_t u + \mathscr{N}u = 0 \text{ in } \Omega \times \boldsymbol{R}^+,$$

and the homogeneous boundary condition

(2) $$\mathscr{B}u(x, t) = 0 \quad \text{on } \partial\Omega \times \boldsymbol{R}^+,$$

where $\mathscr{N}$ is a nonlinear differential operator on x defined exactly afterwards.

When $\mathscr{N}$ is a linear elliptic differential operator on $x \in \Omega$, e.g. $-\Delta$ or Δ^2, the oscillating behavior is well investigated within the framework of the eigenvalue problems. For the linear case we refer to Chapter 5 and 6 of [4]. When $\mathscr{N}$ is nonlinear, it seems that the results have been obtained less compared with the linear case. Cazenave and Haraux have obtained some remarkable results (see [3] and [7]) when $\mathscr{N}$ is semilinear. In [12] results for simpler equations than those of this paper are stated. Besides them we refer to [2] and [9].

In this paper $\mathscr{N}$ is supposed to be more general than that of Cazenave and Haraux. We shall show that there exist different points (x_1, t_1) and (x_2, t_2) in $\Omega \times \boldsymbol{R}^+$ such as $u(x_1, t_1)u(x_2, t_2) < 0$, as is the unsatisfactory result for showing oscillation of u.

Elliptic differential operator of second order is typical of $\mathscr{N}$ and in this case we prescribe the boundary condition to be the homogeneous Dirichlet boundary condition, i.e. $u = 0$ on $\partial\Omega \times \boldsymbol{R}^+$. Besides we can consider $\mathscr{N}$ to be 2^m th order for $m = 2, \dots$ with suitable boundary conditions. For simplicity we shall treat only the $m = 2$ case. Then we prescribe its boundary condition to be one concerned with a supported edge. Here we shall state the second order case in detail.

We don't prove the existence of solutions of initial-boundary value problems satisfying (1), (2) and suitable initial conditions with suitable compatibility conditions, but we suppose the existence of unique global solutions in time (see **A.2** in §3 and **A.5** in §4).

2. Preliminary results. In this section we shall prepare and collect several auxiliary results.

Let $\alpha, \beta, \gamma : \boldsymbol{R} \to \boldsymbol{R}$ be continuous, and α be a positive function of C^1. We define the ordinary differential operator l by

(3) $$l(ly)(t) = (\alpha(t)y'(t))' + \beta(t)y'(t) + \gamma(t)y(t),$$

where $'$ means $\frac{d}{dt}$.

Lemma 2.1. *Let $x(t)$ and $y(t)$ satisfy $(lx)(t) \le 0$ and $(ly)(t) = 0$ in $[t_0, \infty)$ associated with $x(t_0) = y(t_0)$ and $x'(t_0) = y'(t_0)$ for any fixed t_0, respectively. If $y(t) \ge 0$ and $x(t) \ne 0$ for $t \ge t_0$, then $x(t) \le y(t)$ for $t \ge t_0$.*

Proof. Since

$$y(lx) - x(ly) = \{\alpha(x'y - xy')\} + \beta(x'y - xy') \le 0,$$

we get

$$\alpha(t)(x'y - xy')(t)\exp\left(\int_{t_0}^{t} \frac{\beta(s)}{\alpha(s)}\,ds\right) \le \alpha(t_0)(x'y - xy')(t_0) = 0,$$

whence $(x'y - xy')(t) \le 0$. It follows from $(x'y - xy')(t) \le 0$ and $x(t) \ne 0$ that

$$\left(\frac{y(t)}{x(t)}\right)' \ge 0.$$

Hence we have $x(t) \le y(t)$ for $t \ge t_0$. Q.E.D.

In subsequent sections we shall apply the result which assures the existence of zeros of solutions of the differential equation $ly = 0$ to obtain

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