The Maximal Finite Subgroup in the Mapping Class Group of Genus 5

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Abstract: The automorphism groups of compact Riemann surfaces of genus 5 are enumerated by A. Kuribayashi and H. Kimura. Among them, the group of largest order is a group of order 192. The Riemann surface with this automorphism group is unique, and it is realized as the modular curve $X(8)$ of level 8. By utilizing this, we have explicit construction of the finite subgroup of order 192 in the Teichmüller group of genus 5.

0. Introduction. The compactified modular curve $X(8)$ of level 8 corresponding to the principal congruence subgroup $\Gamma(8)$ of $\Gamma(1) = SL_2(\boldsymbol{Z})$ defines a compact complex algebraic curve of genus 5. We are interested in the following problem. Its modulus $[X(8)]$ in the moduli space $\mathcal{M}_5$ of genus 5 curves defines a (singular) point. $\mathcal{M}_5$ is given as a quotient space $\Gamma_5 \backslash \mathcal{T}_5$ of the Teichmüller space $\mathcal{T}_5$ of genus 5 by the Teichmüller group Γ_5 of genus 5. Let $[X(8)]^{\sim}$ be a point of $\mathcal{T}_5$ corresponding to a marking $\beta : \pi_1(X(8), *) \simeq \pi_5$, here π_5 is the surface group of genus 5. Then by a Theorem of Kerckhoff ([1]), the stabilizer of $[X(8)]^{\sim}$ in Γ_5 is isomorphic to the automorphism group $\mathrm{Aut}(X(8)) \cong SL_2(\boldsymbol{Z}/8\boldsymbol{Z})/\{\pm 1\}$. Our problem is to give an explicit description of this stabilizer in $\Gamma_5 = \mathrm{Out}^+(\pi_5)$ in terms of canonical basis of π_5. The same problem for the Klein curve $X(7)$ of genus 3 have been solved by Matsuura using different ideas. ([5])

1. Some general facts. First we briefly describe the well-known construction of the canonical generators in the fundamental group of compact Riemann surface $X_\Gamma = \Gamma \backslash \mathfrak{H}^*$ corresponding to a Fuchsian group of first kind $\Gamma \subset SL_2(\boldsymbol{R})$ ([3]). We are interested in the case when the action of Γ on $\mathfrak{H}$ is fixed-point free. Choose a base point $\Gamma x_0 \in X_\Gamma$, take as a fundamental domain of X_Γ the domain

$$\mathfrak{D} = \bigcap_{\gamma \in \Gamma} \{x \in \mathfrak{H} \mid d(x, x_0) \leq d(x, \gamma x_0)\},$$

where d is $SL_2(\boldsymbol{R})$-invariant metric on $\mathfrak{H}$. Choose an orientation from left to right on the boundary of $\mathfrak{D}$. Each side a of $\mathfrak{D}$ has its conjugate a^{-1}, let $\gamma_a \in \Gamma$ be a map $a \rightarrow a^{-1}$. Denote by $\delta(a)$, the homotopy class of the loop $\delta_1\delta_2$, where δ_1 is a path from x_0 to the endpoint of a and δ_2 is a bath from initial point of a^{-1} to x_0. Then for any relation $\Pi\, a_i^{\pm 1} = 1$ among boundary sides we have $\Pi\, \delta(a_i^{\pm 1}) = 1$ with the same exponents. Thus, we have $\delta(a^{-1}) = \delta(a)^{-1}$. There is another important relation between our loops: for a vertex P of $\mathfrak{D}$ let $a(P)$ be the boundary side starting at P, denote $\sigma(P) = \gamma_{a(P)}(P)$. The cycle of vertex P is a finite set of vertices $\{\sigma^n(P) \mid n \in \boldsymbol{N}\}$. When the cycle of P is $\{P, \sigma(P), \ldots, \sigma^k(P)\}$, we have a relation $\Pi_{i=0}^{k} \delta(a(\sigma^i(P))) = 1$. After eliminating these relations from the fundamental relation, we will get a relation in exactly $2g$ loops, which generate the fundamental group $\pi_1(X_\Gamma, \Gamma x_0)$, here $g = \mathrm{genus}(X_\Gamma)$.

Suppose that, in the fundamental relation two sides a, b and their conjugates a^{-1}, b^{-1} occur in the order $\ldots a \ldots b \ldots a^{-1} \ldots b^{-1} \ldots$. That is , we can write the fundamental relation as $aWbXa^{-1}Yb^{-1}Z = 1$, where W, X, Y, Z are blocks of sides. Firstly, we denote $e = WbX$, our relation transforms to $aea^{-1}YXe^{-1}WZ = 1$ (gluing b on b^{-1}), secondly denote $d = X^{-1}Y^{-1}a$ then, we get a relation $ded^{-1}e^{-1}WZYX = 1$ (gluing a on a^{-1}). After g times repetitions of this procedure we find a generator system with relation $\Pi_{i=1}^{g} [d_i, e_i] = 1$, here $[a, b]$ is the commutator $aba^{-1}b^{-1}$. ([3] section 7.4)

Let now $x_1, x_2 \in \mathfrak{H}^*$ be two points such that $\Gamma x_1 = \Gamma x_2 = \Gamma x_0$. Then the path δ connecting x_1, x_2 in $\mathfrak{H}^*$ defines a closed path on $X_\Gamma(\mathbf{C})$, therefore its homotopy class in $\pi_1(X_\Gamma, \Gamma x_0)$ can be expressed in terms of our canonical generators. The following simple argument give us one such expression. Assume that, δ intersects with the