McKay Correspondence and Hilbert Schemes*)

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Introduction. A particular case in the superstring theory where a finite group G acts upon the target Calabi-Yau manifold M in the theory seems to attract both physicists' and mathematician's attention from various viewpoints. In order to obtain a correct conjectural formula of the Euler number of a smooth resolution of the quotient space M/G, physicists were led to define the following *orbifold Euler characteristic* [2], [3]

$$\chi(M, G) = \frac{1}{|G|}\sum_{gh=hg}\chi(M^{\langle g,h\rangle}),$$

where the summation runs over all the pairs g, h of commuting elements of G, and $M^{\langle g,h\rangle}$ denotes the subset of M of all the points fixed by both of g and h. Then a conjecture of Vafa [2], [3] can be stated in mathematical terms as follows.

Vafa's formula-conjecture. *If a complex manifold M has trivial canonical bundle and if M/G has a (nonsingular) resolution of singularities $\widetilde{M/G}$ with trivial canonical bundle, then we have $\chi(\widetilde{M/G}) = \chi(M, G)$.*

In the special case where $M = A^n$ an n-dimensional affine space, $\chi(M, G)$ turns out to be the number of conjugacy classes, or equivalently the number of equivalence classes of irreducible G-modules. If $n = 2$, then the formula is therefore a corollary to the classical McKay correspondence between the set of exceptional irreducible divisors and the set of equivalence classes of irreducible G-modules [13].

If $n = 3$, then the existence of the above resolution as well as Vafa's formulae is known by the efforts of mathematicians [14], [17], [12], [18], [7], [8], [9], [19]. Except in these cases Vafa's formula is known to be true only in a few cases [6], for instance the case where G is a symmetry group S_m of m letters for $n = 2m$ an arbitrary even integer [5] [15]. In this case $M/G = \mathrm{Symm}^m(A^2)$ and $\widetilde{M/G} = \mathrm{Hilb}^m(A^2)$ as we will see soon. A generalization of the classical McKay correspondence to an arbitrary n

is also known as an Ito-Reid (bijective) correspondence between the set of irreducible exceptional divisors in $\widetilde{M/G}$ and the set of certain conjugacy classes called junior ones [11].

In the present article we will report an interesting return-path from the case where S_n acts on A^{2n} to the two dimensional case with a different G. The analysis of the case leads us to a natural explanation for the classical McKay correspondence mentioned above. We will explain this more precisely in what follows.

Let $\mathrm{Symm}^n(A^2)$ ($\simeq \mathrm{Chow}^n(A^2)$) be the n-th symmetric product of A^2, that is by definition, the quotient of n-copies A^{2n} of A^2 by the natural action of the symmetry group S_n of n letters. Let $\mathrm{Hilb}^n(A^2)$ be the Hilbert scheme of A^2 parametrizing all the 0-dimensional subschemes of length n. By [1] [4] $\mathrm{Hilb}^n(A^2)$ is a smooth resolution of $\mathrm{Symm}^n(A^2)$ with a holomorphic symplectic structure and trivial canonical bundle.

Let G be an arbitrary finite subgroup of $SL(2, \mathbf{C})$. The group G operates on A^2 so that it operates upon both $\mathrm{Hilb}^n(A^2)$ and $\mathrm{Symm}^n(A^2)$ canonically. Now we consider the particular case where n is equal to the order of G. Then it is easy to see that the G-fixed point set $\mathrm{Symm}^n(A^2)^G$ in $\mathrm{Symm}^n(A^2)$ is isomorphic to the quotient space A^2/G. The G-fixed point set $\mathrm{Hilb}^n(A^2)^G$ in $\mathrm{Hilb}^n(A^2)$ is always nonsingular, but can be disconnected and not equidimensional. There is however a unique irreducible component of $\mathrm{Hilb}^n(A^2)^G$ dominating $\mathrm{Symm}^n(A^2)^G$, which we denote by $\mathrm{Hilb}^G(A^2)$. $\mathrm{Hilb}^G(A^2)$ is roughly speaking the Hilbert scheme parametrising all the G-orbits of length $|G|$. Since $\mathrm{Hilb}^G(A^2)$ inherits a holomorphic symplectic structure from

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