Euler's Discretization Revisited

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1. Introduction. It is well known that there are some kinds of scalar ordinary differential equations of which corresponding difference equations (Euler's discretization) are chaotic in the sense of Li-Yorke [1]. This work was done by Yamaguti-Matano in [2]. In this paper, it will be shown that the globally asymptotically stable differential equation which has only one stable equilibrium point is apt to turn into chaos in the corresponding difference equation under some conditions. Moreover, we will give an example such that for any small Δt the difference equation is chaotic.

2. Euler's discretization and Yamaguti–Matano theorem. For the scalar ordinary differential equation

$$\frac{du}{dt} = f(u), \tag{1}$$

Euler's discretization of (1) is defined as follows:

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n), \text{ so } x_{n+1} = x_n + \Delta t f(x_n).$$

Let Euler's difference equation $F_{\Delta t}(x)$ be

$$F_{\Delta t}(x) = x + \Delta t f(x), \tag{2}$$

then the next theorem is obtained.

Theorem (Yamaguti-Matano). Assume $f(u)$ holds condition ($*$).

$$(*)\begin{cases} (i)\, f(u) \text{ is continuous in } R^1 \\ (ii)\, f(0) = f(\bar{u}) = 0\ (\exists\, \bar{u} > 0) \\ (iii)\, f(u) > 0 \qquad (0 < \forall\, u < \bar{u}) \\ (iv)\, f(u) < 0 \qquad (\bar{u} < \forall\, u < K) \\ \qquad\qquad\qquad K \text{ is a constant } (\leq +\infty) \end{cases}$$

Then,

(i) there exists a positive constant c_1 such that for any $\Delta t > c_1$ the difference equation (2) is chaotic in the sense of Li-Yorke.

(ii) Suppose in addition that $K = +\infty$; then there exists another constant c_2, $0 < c_1 < c_2$, such that for any $0 \leq \Delta t \leq c_2$ the map $F_{\Delta t}$ has an invariant finite interval $[0, \alpha_{\Delta t}]$ (i. e., $F_{\Delta t}$ maps $[0, \alpha_{\Delta t}]$ into itself) with $\alpha_{\Delta t} > \bar{u}$. Moreover, when $c_1 < \Delta t \leq c_2$, the above-mentioned chaotic phenomenon occurs in this invariant interval.

In this theorem, (1) have two equilibrium points and chaotic phenomenon occurs around the stable equilibrium point $\bar{u}$. But there are the differential equations with only one stable equilibrium point which turn into chaos. Now we consider three cases as follows: for $u < 0$, $f(u)$ is

$$\begin{cases} \text{Type } A;\ \text{bounded} \\ \text{Type } B;\ O((-u)^{\alpha}) \ (u \to -\infty)\ (0 < \exists\, \alpha < 1) \\ \text{Type } C;\ O((-u)^{1}) \ (u \to -\infty). \end{cases}$$

It will be shown that for each case chaos occurs under some conditions.

3. Type A. From now on, assume $f(u)$ holds following conditions ($**$).

$$(**)\begin{cases} (i)\, f(u) \text{ is continuous in } R^1 \\ (ii)\, f(0) = 0 \\ (iii)\, f(u) > 0\ (\forall\, u < 0) \\ (iv)\, f(u) < 0\ (\forall\, u > 0) \end{cases}$$

Theorem A. Assume $f(u)$ holds ($**$) and the next conditions.

$$\begin{cases} (v)\, f(u) \leq M \qquad (\forall\, u < 0) \\ \qquad\qquad M \text{ is a constant}(< +\infty) \\ (vi)\, f(c_0) = -2M\ (\exists\, c_0 > 0) \end{cases}$$

Then there exists a positive T such that for any $\Delta t \geq T$, $F_{\Delta t}$ is chaotic in the sense of Li-Yorke in an invariant finite interval.

Proof of Theorem A. To prove chaos in the sense of Li-Yorke, it is enough to show the existence of an a such that $b = F_{\Delta t}(a)$, $c = F_{\Delta t}(b)$ and $d = F_{\Delta t}(c)$ satisfy $d \leq a < b < c$. Set

$$T = \min_{x<0} \frac{c_0 - x}{f(x)} \left(\geq \frac{c_0}{M}\right).$$

$(c_0 - x)/f(x)$ is continuous and positive in $x < 0$, and also bounded from below because $\lim_{x\to-0} \frac{c_0 - x}{f(x)} = +\infty$, $\lim_{x\to-\infty} \frac{c_0 - x}{f(x)} = +\infty$. Hence T exists. And for any $\Delta t \geq T$ there is a b (< 0) such that $(c_0 - b)/f(b) = \Delta t$ holds. For c and d, $c = F_{\Delta t}(b) = b + \Delta t f(b) = c_0$ (> 0), $d = F_{\Delta t}(c) = c + \Delta t f(c) = c_0 - 2M\Delta t$. On the other hand, since $F_{\Delta t}(b) = b + \Delta t f(b) > b$ and $F_{\Delta t}(b - \Delta t M) = b + \Delta t(f(b - \Delta t M) - M) \leq b$, there exists an a which satisfies $F_{\Delta t}(a) = b$ in