# Minor Summation Formula of Pfaffians and Schur Function Identities

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**1. Introduction.** In the paper [1], we exploited a minor summation formula of Pfaffians. The prototype of this formula is found in [6]. The merit of our formula is that, by taking various antisymmetric matrices, we obtain considerably various formulas on the summations of minors of a given rectangular matrix. Our motivation was in the use of the enumerative combinatorics and combinatorial representation theory. (See [9].) We are expecting the utility of this formula on various objects in this area. Particularly we think that the applications on the Schur function identities are important and we studied them intensively in [2]. There we obtained new proof of the formulas which are usually called Littlewood's formulas. Typical examples of Littlewood's formulas are the followings.

$$\sum_{\lambda=(\alpha|\alpha+1)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i<j\le m} (1-x_ix_j), \tag{1.1}$$

$$\sum_{\lambda=(\alpha|\alpha)} (-1)^{\frac{|\lambda|}{2}+p(\lambda)} s_\lambda(x_1,\dots,x_m) = \prod_{i=1}^{m} (1-x_i) \prod_{1\le i<j\le m} (1-x_ix_j), \tag{1.2}$$

$$\sum_{\lambda=(\alpha+1|\alpha)} (-1)^{\frac{|\lambda|}{2}} s_\lambda(x_1,\dots,x_m) = \prod_{1\le i<j\le m} (1-x_ix_j). \tag{1.3}$$

(See [4].) For the notation see Section 2. In this paper we state some new results which are obtained after [2]. The method we use owes to [2], but we develop the method and exploit certain new identities which involve both the Schur functions and Čebyšev's polynomials. The main results of this paper are Theorems 3.1, 3.2 and 3.3. In the process of deriving these identities, the argument on the relation between (Sato's) Maya diagram and Murnaghan-Nakayama's formula on Young diagram has a crucial role.

**2. Basic notation and a summation formula.** In the paper [1] we exploited a minor summation formula of Pfaffians. Now we briefly review this formula.

Let $r$, $m$, $n$ be positive integers such that $r \le m, n$. Let $T$ be an arbitrary $m$ by $n$ matrix. For two sequences $\boldsymbol{i} = (i_1, \dots, i_r)$ and $\boldsymbol{k} = (k_1, \dots, k_r)$, let $T^{\boldsymbol{i}}_{\boldsymbol{k}} = T^{i_1\dots i_r}_{k_1\dots k_r}$ denote the submatrix of $T$ obtained by picking up the rows and columns indexed by $\boldsymbol{i}$ and $\boldsymbol{k}$, respectively.

Assume $m \le n$ and let $B$ be an arbitrary $n$ by $n$ antisymmetric matrix, that is, $B = (b_{ij})$ satisfies $b_{ij} = -b_{ji}$. As long as $B$ is a square antisymmetric matrix, we write $B_{\boldsymbol{i}} = B_{i_1\dots i_r}$ for $B^{\boldsymbol{i}}_{\boldsymbol{i}} = B^{i_1\dots i_r}_{i_1\dots i_r}$ in abbreviation. One of the main result in [1] is the following theorem. (See Theorem 1 of [1].)

**Theorem 2.1.** *Let $m \le n$ and $T = (t_{ik})$ be an arbitrary $m$ by $n$ matrix. Let $m$ be even and $B = (b_{ik})$ be any $n$ by $n$ antisymmetric matrix with entries $b_{ik}$. Then*

$$\sum_{1\le k_1<\cdots<k_m\le n} \mathrm{pf}(B_{k_1\dots k_m})\det(T^{1\dots m}_{k_1\dots k_m}) = \mathrm{pf}(Q), \tag{2.1}$$

*where $Q$ is the $m$ by $m$ antisymmetric matrix defined by $Q = TB^tT$, i.e.*

$$Q_{ij} = \sum_{1\le k<l\le n} b_{kl}\det(T^{ij}_{kl}), \quad (1\le i,j\le m). \tag{2.2}$$

We regard the Pfaffian $\mathrm{pf}(B_{\boldsymbol{k}})$ as certain "weights" of the subdeterminants $\det(T^{1\dots m}_{k_1\dots k_m})$. By changing this antisymmetric matrix we obtain a considerably wide variation of the minor summation formula.

Now we review some basic notation. The reader can find these notation in [5]. A weakly decreasing sequence of nonnegative integers $\lambda := (\lambda_1, \cdots, \lambda_m)$ with $\lambda_1 \ge \cdots \ge \lambda_m \ge 0$ is called a *partition* of $|\lambda| = \lambda_1 + \cdots + \lambda_m$. The partition

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