The Restriction of $A_{\mathfrak{q}}(\lambda)$ to Reductive Subgroups II

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§ 1. Introduction. In this paper we continue the investigation of the restriction of irreducible unitary representations of real reductive groups, with emphasis on the discrete decomposability. We recall that a representation π of a reductive Lie group G on a Hilbert space V is G-*admissible* if (π, V) is decomposed into a discrete Hilbert direct sum with finite multiplicities of irreducible representations of G. The same terminology is used for a $(\mathfrak{g}, K)$-module on a pre-Hilbert space, if its completion is G-admissible.

Let H be a reductive subgroup of a real reductive Lie group G, and (π, V) an irreducible unitary representation of G. The restriction $(\pi_{|H}, V)$ is decomposed uniquely into irreducible unitary representations of H, which may involve a continuous spectrum if H is noncompact. In [5],[6], we have posed a problem to single out the triplet (G, H, π) such that the restriction of $(\pi_{|H}, V)$ is H-admissible, together with some application to harmonic analysis on homogeneous spaces. The purpose of this paper is to give a new insight of such a triplet (G, H, π) from view points of algebraic analysis. In particular, we will give a sufficient condition on the triplet (G, H, π) for the H-admissible restriction as a generalization of [5],[6] to arbitrary H, and also present an obstruction for the H-admissible restriction.

§ 2. A sufficient condition for discrete decomposability. Let K be a compact Lie group. We write $\mathfrak{k}_0$ for the Lie algebra of K, and $\mathfrak{k}$ for its complexification. Analogous notation is used for other groups. Take a Cartan subalgebra $\mathfrak{t}_0^c$ of $\mathfrak{k}_0$. The weight lattice L in $\sqrt{-1}(\mathfrak{t}_0^c)^*$ is the additive subgroup of $\sqrt{-1}(\mathfrak{t}_0^c)^*$ consisting of differentials of the weights of finite dimensional representations of K. Let $\bar{C} \subset \sqrt{-1}(\mathfrak{t}_0^c)^*$ be a dominant Weyl chamber. We write K_0 for the identity component of K, and $\widehat{K_0}$ for the unitary dual of K_0. The Cartan-Weyl theory of finite dimensional representations establishes a bijection:

$$L \cap \bar{C} \xrightarrow{\sim} \widehat{K_0},\ \lambda \mapsto F(K_0, \lambda).$$

Suppose X is a K-module (possibly, of infinite dimension) which carries an algebraic action of K. The K_0-multiplicity function of X is given by

$$m \equiv m_X : L \cap \bar{C} \to \boldsymbol{N} \cup \infty,$$
$$m(\lambda) := \dim \operatorname{Hom}_{K_0}(F(K_0, \lambda), X).$$

The asymptotic K-support $T(X) \subset \bar{C}$ was introduced in [3] as follows:

$S(X) := \{\lambda \in L \cap \bar{C} : m_X(\lambda) \neq 0\}$,

$T(X) := \{\lambda \in \bar{C} : V \cap S(X)$ is not relatively compact for any open cone V containing $\lambda\}$.

Hereafter we assume a growth condition on m_X: there are constants $A, R > 0$ such that

(2.1) $\quad m_X(\lambda) \le A \exp(R|\lambda|)$ for any $\lambda \in L \cap \bar{C}$.

This condition assures that the character of the representation X is a hyperfunction on K, whose singularity spectrum we can estimate in terms of $T(X)$.

Suppose H is a closed subgroup of K. Let $\mathrm{pr}_{K\to H} : \mathfrak{k}^* \to \mathfrak{h}^*$ be the projection dual to the inclusion of Lie algebras $\mathfrak{h} \hookrightarrow \mathfrak{k}$. Put $\mathfrak{h}^\perp := \operatorname{Ker}(\mathrm{pr}_{K\to H} : \mathfrak{k}^* \to \mathfrak{h}^*)$. We set

(2.2) $\quad \bar{C}(\mathfrak{h}) := \bar{C} \cap \mathrm{Ad}^*(K)\mathfrak{h}^\perp \subset \sqrt{-1}(\mathfrak{t}_0^c)^*$.

Note that $\bar{C}(\mathfrak{k}) = \{0\}$ and $\bar{C}(0) = \bar{C}$.

Theorem 2.3. *Let X be a K-module satisfying* (2.1). *If a closed subgroup H of K satisfies*

$$T(X) \cap \bar{C}(\mathfrak{h}) = \{0\},$$

then the restriction $X_{|H}$ is H-admissible.

Now, let us apply Theorem (2.3) to some standard $(\mathfrak{g}, K)$-modules. Suppose that G is a real reductive linear Lie group and that K is a maximal compact subgroup of G. A dominant element $a \in \sqrt{-1}\,\mathfrak{t}_0^c$ defines a θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, where $\mathfrak{l}, \mathfrak{u}$ are the sum of eigenspaces of $\mathrm{ad}(a)$ with 0, positive eigenvalues, respectively. Let L be the centralizer of a in G. Zuckerman introduced the cohomological parabolic induction $\mathcal{R}_{\mathfrak{q}}^j \equiv (\mathcal{R}_{\mathfrak{q}}^{\mathfrak{g}})^j$ $(j \in \boldsymbol{N})$, which is a covariant functor from the category of metaplectic $(\mathfrak{l}, (L \cap K)^{\sim})$-modules to that of $(\mathfrak{g}, K)$-modules, as a generalization of the Borel-Weil-Bott con-