On Arithmetic of Certain Matrix Algebras

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1. Introduction. Let $GL(n, \boldsymbol{C})$ be the group of all invertible matrices of degree n with entries in the complex number field $\boldsymbol{C}$. An element A in $GL(n, \boldsymbol{C})$ is called *regular* if the centralizer T of A in $GL(n, \boldsymbol{C})$ forms a maximal split torus of the reductive group $GL(n, \boldsymbol{C})$. By $GL(n, \boldsymbol{Z})$ we denote the modular group of degree n over the ring of integers $\boldsymbol{Z}$. Let ζ be a regular element in $GL(n, \boldsymbol{Z})$ and $R = \boldsymbol{Z}[\zeta]$ the ring generated by ζ over $\boldsymbol{Z}$. We shall define as follows the ideal class semigroup $\boldsymbol{G}$ of R. An ideal $\mathfrak{a}$ of R is *nonsingular* if the index $(R : \mathfrak{a})$ of additive subgroup $\mathfrak{a}$ of R is finite. $N\mathfrak{a} = (R : \mathfrak{a})$ is called *the norm* of $\mathfrak{a}$. Let $\boldsymbol{Q}[\zeta]$ be the ring generated by ζ over the rational number field $\boldsymbol{Q}$. A R-submodule $\mathfrak{a}$ of $\boldsymbol{Q}[\zeta]$ is called a *fractional ideal* if there exists an invertible element α in $\boldsymbol{Q}[\zeta]$ such that $\alpha\mathfrak{a}$ is a nonsingular ideal of R. Let $\boldsymbol{A}$ be the set of all fractional ideals of R. $\boldsymbol{A}$ is a semigroup with the canonical multiplication. The group $\boldsymbol{Q}[\zeta]^{\times}$ of all invertible elements in $\boldsymbol{Q}[\zeta]$ acts on the set $\boldsymbol{A}$. We classify $\boldsymbol{A}$ into the orbit classes under $\boldsymbol{Q}[\zeta]^{\times}$. The set of these classes forms a semigroup $\boldsymbol{G}$ which will be called *the ideal class semigroup* of R (cf. [17]).

We recall that these algebras $R = \boldsymbol{Z}(\zeta)$ and the ideal class semigroups $\boldsymbol{G}$ of these algebras have already been studied in [14],[22], where a bijective mapping of $\boldsymbol{G}$ to the set of conjugacy classes $G_Z(f)/GL(n, \boldsymbol{Z})$ given in the following sense. Let $f(X)$ be the characteristic polynomial of ζ (which has only simple roots as ζ is regular). $G_Z(f)$ is the set of elements of $GL(n, \boldsymbol{Z})$ with the chracteristic polynomial $f(X)$, which is decomposed into $GL(n, \boldsymbol{Z})$ orbit classes, the action of an element of $GL(n, \boldsymbol{Z})$ being adjoint action. $G_Z(f)/GL(n, \boldsymbol{Z})$ means the orbit space. The finiteness of the space $G_Z(f)/GL(n, \boldsymbol{Z})$ has been proved by [19] ,[23](cf. also the related works [15], [21],[12] and [8]).

The purpose of this note is to develop the arithmetic of R and to introduce in particular Dirichlet series which can be utilized to calculate $|\boldsymbol{G}|$. The methods we have used in [17] are found here useful. The detailed discussion with proof will appear elsewhere.

We remind that zeta functions of various kinds have been introduced into the study of algebras in the papers [2]-[4], [6], [9]-[11], [13] and [20]. Particularly, Solomon's idea in dealing with group algebras in [20] and its generalization by Bushnell-Reiner [2], [3], concerning semisimple $\boldsymbol{Q}$-algebras, have given suggestions for this paper.

We shall define the norm in the ring $\boldsymbol{Q}[\zeta]$. Let T be the centralizer of ζ in $GL(n, \boldsymbol{C})$. We can choose a subset

$$\Omega = \{\zeta, \zeta', \dots, \zeta^{(n-1)}\}$$

of T satisfying

$$(1.1) \quad \Delta(\zeta) = \prod_{0 \le i < j < n} (\zeta^{(i)} - \zeta^{(j)}) \in GL(n, \boldsymbol{C}).$$

Ω is the set of algebraic conjugates of ζ. By (1.1) we can prove that the characteristic polynomial $f(X)$ of ζ is factorized as

$$(1.2) \quad f(X) = (X - \zeta)(X - \zeta') \cdots (X - \zeta^{(n-1)}).$$

Let α be an element in $\boldsymbol{Q}[\zeta]$ and $p[X]$ a polynomial with degree $< n$ satisfying $\alpha = p(\zeta)$. We define i-th conjugate $\alpha^{(i)}$ by $\alpha^{(i)} = p(\zeta^{(i)})$. The norm $N\alpha$ is defined by

$$N\alpha = \alpha\alpha' \cdots \alpha^{(n-1)}.$$

Finally we shall state the properties of the ring of integers O and of the unit group $\boldsymbol{E}_O$ of $\boldsymbol{Q}[\zeta]$. Bearing in mind that all eigenvalues of ζ are mutually distinct we see that $f(X)$ is decomposed into irreducible divisors

$$f_1(X), f_2(X), \dots, f_g(X)$$

over $\boldsymbol{Z}$ with multiplicity one. We put $h_i(X) = f(X)/f_i(X)$. Then there exist the polynomials $u_1(X), u_2(X), \dots, u_g(X)$ with rational coefficients such that

$$\sum_{i=1}^{g} u_i(X) h_i(X) = 1.$$

We put $e_i = u_i(\zeta) h_i(\zeta)$. Then we have