Commutation Relations of Differential Operators and Whittaker Functions on $Sp_2(\boldsymbol{R})$*)

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§1. As usual we consider an element in the center of the universal enveloping algebra of Lie algebra of a Lie group G as a differential operator on G. Generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \boldsymbol{R})$ are given in [6].

Put

$$H_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ H_2=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},$$

$$X_1=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\end{pmatrix},\ X_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$X_3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ X_4=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$X_{-i}={}^tX_i\ (1\le i\le 4).$$

Then the generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \boldsymbol{R})$ in [6] are

$$\lambda(L_1)=H_1H_1+H_2H_2+6H_1+2H_2 +4X_{-1}X_1+8X_{-4}X_4+4X_{-3}X_3+8X_{-2}X_2,$$

$$\begin{aligned}\lambda(L_2)=\ &16X_{-4}X_{-4}X_4X_4+16X_{-4}X_{-3}X_3X_4\\ &-32X_{-4}X_{-2}X_2X_4+16X_{-4}X_{-2}X_3X_3\\ &+16X_{-4}X_{-1}X_1X_4+8X_{-4}H_1H_2X_4\\ &+8X_{-4}(H_1-H_2)X_1X_3-16X_{-4}X_1X_1X_2\\ &+16X_{-3}X_{-3}X_2X_4+16X_{-3}X_{-2}X_2X_3\\ &+8X_{-3}X_{-1}(H_1-H_2)X_4+4X_{-3}H_2H_2X_3\\ &+8X_{-3}(H_1+H_2)X_1X_2+16X_{-2}X_{-2}X_2X_2\\ &-16X_{-2}X_{-1}X_{-1}X_4+8X_{-2}X_{-1}(H_1+H_2)X_3\\ &+16X_{-2}X_{-1}X_1X_2-8X_{-2}H_1H_2X_2\\ &+4X_{-1}H_1H_1X_1+H_1H_1H_2H_2-16X_{-4}H_1X_4\\ &+32X_{-4}H_2X_4+32X_{-4}X_1X_3+32X_{-3}X_{-1}X_4\\ &-8X_{-3}H_1X_3+16X_{-3}X_1X_2+16X_{-2}X_{-1}X_3\\ &-16X_{-2}(H_1+H_2)X_2+24X_{-1}H_1X_1\\ &+2H_1H_1H_2+6H_1H_2H_2-48X_{-4}X_4\\ &-24X_{-3}X_3-48X_{-2}X_2+24X_{-1}X_1-2H_1H_1\\ &+12H_1H_2+6H_2H_2-12H_1+12H_2.\end{aligned}$$

We can find the generators of the centers of the universal enveloping algebras of the Lie algebras of classical groups by [6], [3]. The generators of the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}=\mathfrak{sl}(4, \boldsymbol{R})$ are f_2, f_4, f_6 in [3, §13, no. 4, (VI), p. 189]. The polynomial functions f_2, f_4, f_6 on $\mathfrak{g}$ are the coefficients of the characteristic polynomial of the identity representation of $\mathfrak{g}$. Identifying the dual of $\mathfrak{g}$ with $\mathfrak{g}$ and applying a symmetrizer Λ such that

$$\Lambda(X_1X_2\dots X_n)=\sum_{\sigma\in\mathfrak{S}_n}X_{\sigma(1)}X_{\sigma(2)}\dots X_{\sigma(n)}\quad (X_i\in\mathfrak{g})$$

on the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ to f_2, f_4, f_6, we get the generators $\beta_2=\Lambda(f_2)$, $\beta_3=\Lambda(f_4)$, $\beta_4=\Lambda(f_6)$ of the center of $U(\mathfrak{g})$.

§2. We define the Weil representation r_n of $Sp_2(\boldsymbol{R})$ on $\mathcal{S}(V_n\times V_n)$, $V=V_n=M_{n,2}(\boldsymbol{R})$ by putting

$$r_n\begin{pmatrix}E&X\\0&E\end{pmatrix}f\begin{pmatrix}X_1\\X_2\end{pmatrix}=\exp(2\pi i\operatorname{tr}(X\,{}^tX_1X_2))\,f\begin{pmatrix}X_1\\X_2\end{pmatrix},$$

$$r_n\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix}f\begin{pmatrix}X_1\\X_2\end{pmatrix}=(\det A)^nf\begin{pmatrix}X_1A\\X_2A\end{pmatrix},$$

$$r_n\begin{pmatrix}0&E\\-E&0\end{pmatrix}f\begin{pmatrix}X_1\\X_2\end{pmatrix}=\int_{V_n}\int_{V_n}\exp(2\pi i\operatorname{tr}({}^tY_1{}^tX_2+{}^tY_2X_1))\,f\begin{pmatrix}Y_1\\Y_2\end{pmatrix}dY_1dY_2$$

for $f\in\mathcal{S}(V_n\times V_n)$, $X={}^tX\in M_{2,2}(\boldsymbol{R})$, $A\in M_{2,2}(\boldsymbol{R})$, $E=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $X_1\in V_n$, $X_2\in V_n$.

Put $G_1=SL(2,\boldsymbol{R})$, $G_3=SL(4,\boldsymbol{R})$. Then we can define representations ρ_2, ρ_3 of $G_1\times G_1$, G_3 on $\mathcal{S}(V_2\times V_2)$, $\mathcal{S}(V_3\times V_3)$ in the following manner. First we define linear mappings σ_1, σ_3 by

$$\sigma_1(X)=\begin{pmatrix}a&d\\b&-c\end{pmatrix}\text{ for }X={}^t(a\ b\ c\ d)\in M_{4,1}(\boldsymbol{R})$$

and

*) Dedicated to Professor Hideo Shimizu on his sixtieth birthday.