Triangles and Elliptic Curves. VI

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This is a continuation of the series of papers [1] each of which will be referred to as (I), (II), (III), (IV), (V) in this paper. By a *real* triangle we shall mean an element of the following set:

(0.1) $Tr = \{t = (a, b, c) \in \boldsymbol{R}^3,\ 0 < a < b + c,\ 0 < b < c + a,\ 0 < c < a + b\}.$

For each $t \in Tr$, set $s = s(t) = \frac{1}{2}(a + b + c)$.

One sees easily that

(0.2) $Tr = \{t = (a, b, c) \in \boldsymbol{R}^3,\ 0 < a, b, c < s\}.$

As in (I), we associate an elliptic curve E_t to $t \in Tr$:

(0.3) $E_t : y^2 = x^3 + P_t x^2 + Q_t x$

where

(0.4) $P_t = \frac{1}{2}(a^2 + b^2 - c^2),$

(0.5) $Q_t = -s(s - a)(s - b)(s - c) = -(\text{area of } t)^2.$

In this paper, we shall describe isomorphisms (over $\boldsymbol{R}$) among elliptic curves (0.3) in terms of relations among triangles (0.1).

§1. Basic facts. Let k be a field of characteristic not 2. Consider an elliptic curve of the form:

(1.1) $y^2 = x^3 + Px^2 + Qx,\ P, Q \in k.$

Referring to the standard notation of Weierstrass equations ([2], Chapter III, §1), we have

(1.2) $a_1 = a_3 = a_6 = 0,\ a_2 = P,\ a_4 = Q,$

(1.3) $b_2 = 4P,\ b_4 = 2Q,\ b_6 = 0,\ b_8 = -Q^2,$

(1.4) $c_4 = 16(P^2 - 3Q),\ c_6 = -32P(2P^2 - 9Q),$

(1.5) $\Delta = 16Q^2(P^2 - 4Q) \neq 0,$

(1.6) $j = c_4^{\,3}/\Delta = 2^8(P^2 - 3Q)^3/(Q^2(P^2 - 4Q)).$

Now let $k = \boldsymbol{R}$. Inspired by (0.5) for triangles, *we shall focus our attention on elliptic curves* (1.1) *with* $Q < 0$. Thus we have, from (1.4), (1.5), (1.6),

(1.7) $c_4 > 0,\ \Delta > 0,\ j > 0$

and

(1.8) $\operatorname{sign}(c_6) = -\operatorname{sign}(P),\quad c_6 = 0 \Leftrightarrow P = 0 \Leftrightarrow j = 1728.$

From now on, for a real number $a > 0$, we assume that $\sqrt{a} > 0$. We put

(1.9) $M = \frac{1}{2}(P + \sqrt{P^2 - 4Q}),\quad N = \frac{1}{2}(P - \sqrt{P^2 - 4Q}).$

Since $M - N = \sqrt{P^2 - 4Q} > 0$ and $MN = Q < 0$, we have

(1.10) $M > 0,\ N < 0.$

From (1.1), (1.9), it follows that

(1.11) $y^2 = x^3 + Px^2 + Qx = x(x + M)(x + N).$

Now, we introduce a quantity

(1.12) $\lambda = N/M < 0.$

Since the elliptic curve (1.11) is isomorphic (over $\boldsymbol{C}$) to the Legendre form $y^2 = x(x - 1)(x - \lambda)$, we obtain

(1.13) $j = 2^8(\lambda^2 - \lambda + 1)^3/(\lambda^2(\lambda - 1)^2).$

Next, we put

(1.14) $\rho = -\frac{1}{2}(\lambda + \lambda^{-1}) = 1 - (P^2/2Q) \geq 1.$

Finally, following [3], Chapter V, §2, define a quantity γ:

(1.15) $\gamma = \begin{cases} \operatorname{sign}(c_6), & \text{if } j \neq 1728 \text{ (i.e., if } c_6 \neq 0) \\ \operatorname{sign}(c_4), & \text{if } j = 1728 \text{ (i.e., if } c_6 = 0). \end{cases}$

In view of (1.8), we have

(1.16) $\gamma = \begin{cases} 1, & \text{if } P \leq 0 \\ -1, & \text{if } P > 0. \end{cases}$

(1.17) **Proposition.** *Let* E, E' *be elliptic curves over* $\boldsymbol{R}$ *of the form* $E : y^2 = x^3 + Px^2 + Qx$, $E' : y^2 = x^3 + P'x^2 + Q'x$ *with* $Q, Q' < 0$. *Let* j, λ, ρ, γ (*resp.* $j', \lambda', \rho', \gamma'$) *be quantities* (1.6), (1.12), (1.14), (1.15) *for* E (*resp.* E'). *Then we have*

$E \cong E'$ *over* $\boldsymbol{R} \Leftrightarrow \rho = \rho'$ *and* $\operatorname{sign} P = \operatorname{sign} P'$.

Proof. First of all, we know ([3], Chapter V, §2) that

(1.18) $E \cong E'$ *over* $\boldsymbol{R} \Leftrightarrow j = j'$ *and* $\gamma = \gamma'$.

Now since λ, λ' are both < 0, we have

$$j' = j \Leftrightarrow \lambda' \in \{\lambda, 1/\lambda, 1 - \lambda, 1/(1 - \lambda), \lambda/(\lambda - 1), (\lambda - 1)/\lambda\} \Leftrightarrow \lambda' \in \{\lambda, 1/\lambda\} \Leftrightarrow \rho' = \rho.$$

Our assertion then follows from these equivalences and (1.16), (1.18). Q.E.D.

(1.19) **Corollary.** *Elliptic curves* $y^2 = x^3 + Qx$, $Q < 0$, *are all isomorphic over* $\boldsymbol{R}$.