Wavelet Transforms Associated to a Principal Series Representation of Semisimple Lie Groups. II

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(Communicated by Kiyosi ITÔ, M. J. A., Sept. 12, 1995)

1. Introduction. Let G be a noncompact connected semisimple Lie group with finite center and $P = MAN$ a parabolic subgroup of G. Let $\pi_\lambda = \mathrm{Ind}_P^G(1 \otimes e^\lambda \otimes 1)$ ($\lambda \in \mathfrak{a}_C^*$) denote a principal series representation of G and $(\pi_\lambda, L^2(\bar{N}, e^{-2\Re\lambda(H(\bar{n}))}d\bar{n})$ $(\bar{N} = \theta(N))$ the noncompact picture of π_λ. Let σ_ω denote an irreducible unitary representation of $\bar{N}$ corresponding to $\omega \in \bar{\mathfrak{n}}_C^*$ and (S, ds) a subset of MA with measure ds. In the previous paper [3] we supposed that there exists a $\phi \in \mathcal{S}'(\bar{N})$ satisfying the following admissible condition: for all $\omega \in V_T'$

(*i*) $\sigma_\omega(\phi)\sigma_\omega(\phi)^* = n_\phi(\omega)I$,

(*ii*) $0 < \int_S n_\phi(Ad(s)\omega)ds = c_{S,\phi} < \infty$,

where $c_{S,\phi}$ is independent of ω (see [3] for the notations). Then for all such ϕ we can deduce the inversion formula:

$$f(x) = c_{S,\phi}^{-1} \int\int_{\bar{N}\times S} \langle f, \pi_{-i\rho}(\bar{n}s)\phi \rangle \cdot \pi_{-i\rho}(\bar{n}s)\phi(x)d\bar{n}ds \quad \text{for all } f \in \mathcal{S}(\bar{N}),$$

where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\bar{N})$. A number of well-known examples of wavelet transforms arises from this scheme through the explicit form of ϕ. However, in the case of $G = SL(n+2, \boldsymbol{R})$ $(n \geq 1)$ and $\bar{N} \cong H_n$, the $(2n+1)$-dimensional Heisenberg group, the above formula does not cover the three examples constructed by Kalisa and Torrésani (see [4, IV]). Therefore, in order to obtain a widespread application we need to generalize this formula. In this paper we suppose that S is an arbitrary measurable set with map $l: S \to G$ and then we shall consider a distribution vector ϕ in $\mathcal{S}'(\bar{N})$ which depends on $s \in S$.

2. Main theorem. We retain the notations in [3] except that (S, ds) is an arbitrary measurable set with map $l: S \to G$. Let Ψ be a family of $\phi_s \in \mathcal{S}'(\bar{N})$ with parameter $s \in S$. We call the quartet $\mathfrak{A} = (\lambda, S, l, \Psi)$ satisfies the admissible condition if for all $\omega \in V_T'$ and $F \in L^2(\boldsymbol{R}^k)$

$$\int_S \sigma_\omega(\pi_\lambda(l(s)\phi_s))\sigma_\omega(\pi_\lambda(l(s)\phi_s))^* F ds = c_{\mathfrak{A}}F,$$

where σ_ω is realized on $L^2(\boldsymbol{R}^k)$ (see §3) and $c_{\mathfrak{A}}$ is independent of ω.

Theorem 1. *Let* $\mathfrak{A} = (\lambda, S, l, \Psi)$ *satisfy the admissible condition. Then,*

$$f(x) = c_{\mathfrak{A}}^{-1} \int\int_{\bar{N}\times S} \langle f, \pi_\lambda(\bar{n}l(s))\phi_s \rangle \cdot \pi_\lambda(\bar{n}l(s))\phi_s(x)d\bar{n}ds \ \textit{for all}\ f \in \mathcal{S}(\bar{N}).$$

Proof. As shown in [2] it is enough to prove that

$$\int_S \| \langle f, \pi_\lambda(\cdot)\Psi_s \rangle \|_{L^2(\bar{N})}^2 ds = c_{\mathfrak{A}} \| f \|_{L^2(\bar{N})}^2,$$

where $\Psi_s = \pi_\lambda(l(s))\phi_s$. Since $\sigma_\omega(\langle f, \pi_\lambda(\cdot)\Psi_s \rangle) = \sigma_\omega(f)\sigma_\omega(\Psi_s)^*$, it follows from the Plancherel formula for $L^2(\bar{N})$ that

$$\begin{aligned}&\int_S \| \langle f, \pi_\lambda(\cdot)\phi_s \rangle \|_{L^2(\bar{N})}^2 ds \\ &= \int_S\int_{V_T'} \| \sigma_\omega(f)\sigma_\omega(\Psi_s)^* \|_{HS}^2 \mu(\omega)d\omega ds \\ &= \int_{V_T'} \mathrm{tr}\Big(\sigma_\omega(f) \int_S \sigma_\omega(\Psi_s)^*\sigma_\omega(\Psi_s)ds\sigma_\omega(f)^*\Big)\mu(\omega)d\omega \\ &= c_{\mathfrak{A}} \| f \|_{L^2(\bar{N})}^2. \qquad \square\end{aligned}$$

3. Admissible condition. In what follows we assume that

(A0) $l(S) \subset MA$,

and we shall obtain a sufficient condition of $\mathfrak{A} = (\lambda, S, l, \Psi)$ under which $\mathfrak{A}$ is admissible. Let $\mathfrak{q}$ be a polarizing subalgebra for all $\omega \in V_T'$ and Q the corresponding analytic subgroup of $\bar{N}$. We put $k = \mathrm{codim}\mathfrak{q}$, $\chi_\omega(\exp Y) = e^{2\pi i\omega(Y)}$ $(Y \in \mathfrak{q})$, and $\bar{n} = \exp X(\bar{n})\gamma(t(\bar{n}))$ $(X(\bar{n}) \in \mathfrak{q}, t(\bar{n}) \in \boldsymbol{R}^k)$ where $\gamma: \boldsymbol{R}^k \to \bar{N}$ is a cross-section for $Q\backslash\bar{N}$. Then $\sigma_\omega = \mathrm{Ind}_Q^{\bar{N}}(\chi_\omega)$ and it is realized on $L^2(\boldsymbol{R}^k)$ as $\sigma_\omega(\bar{n})F(t) = \chi_\omega(X(\gamma(t)\bar{n}))F(t(\gamma(t)\bar{n}))$ (cf. [1, p.125]). Here we recall that $l(s) \in MA$ and a weak Malcev basis consists of root vectors for (G, A). Thus $Ad(l(s))$ stabilizes Q and $Q\backslash\bar{N}$ respectively. Here we suppose that

(A1) $\mathfrak{q}$ is ideal,

1991 Mathematics Subject Classification. Primary 22E30; Secondary 42C20.