Wavelet Transforms Associated to a Principal Series Representation of Semisimple Lie Groups. I

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1. Introduction. Let G be a locally compact Lie group and π a continuous representation of G on a Hilbert space $\mathscr{H}$. Let $\mathscr{H}_\infty$ denote the space of C^∞-vectors in $\mathscr{H}$, endowed with a natural Sobolev-type topology, and $\mathscr{H}_{-\infty}$ the dual of $\mathscr{H}_\infty$ endowed with the strong topology. We denote the corresponding representation on $\mathscr{H}_{-\infty}$ by the same letter π. Let S be a subset of G and ds a measure on S. A vector $\psi \in \mathscr{H}_{-\infty}$ is said to be S-strongly admissible for π if there exists a positive constant $c_{S,\psi}$ such that

$$(1) \qquad \int_S |\langle f, \pi(s)\psi\rangle_{\mathscr{H}}|^2 ds = c_{S,\psi} \| f \|_{\mathscr{H}}^2 \quad \text{for all } f \in \mathscr{H}_\infty,$$

where $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ and $\|\cdot\|_{\mathscr{H}}$ denote the inner product and the norm of $\mathscr{H}$ respectively. We easily see that $\psi \in \mathscr{H}_{-\infty}$ is S-strongly admissible for π if and only if, as a functional on $\mathscr{H}_\infty$,

$$(2) \qquad f = c_{S,\psi}^{-1} \int_S \langle f, \pi(s)\psi\rangle_{\mathscr{H}} \pi(s)\psi ds \quad \text{for all } f \in \mathscr{H}_\infty.$$

We call $\langle f, \pi(s)\psi\rangle$ the wavelet transform of f associated to (G, π, S, ψ) in the sense that, by specializing (G, π, S, ψ), the above formula yields a group theoretical interpretation of various well-known wavelet transforms. For example, we first let $S = G$, $ds = dg$, a Haar measure of G, and $(\pi, \mathscr{H})$ a square-integrable representation of G, that is, π is an irreducible unitary representation satisfying $0 < \int_G |\langle \phi, \pi(g)\psi\rangle|^2 dg < \infty$ for all ϕ, ψ in $\mathscr{H}$. Then π is a discrete series of G and every $\psi \in \mathscr{H}$ is a G-strongly admissible vector for π (see [3]). The Gabor transform and the Grossmann-Morlet transform correspond to the Weyl-Heisenberg group and the one-dimensional affine group respectively (cf. [7, §3]). Next let H be a closed subgroup of G and π a discrete series of G/H. Then there exists an H-invariant distribution vector $\psi \in \mathscr{H}_{-\infty}$ for which (2) holds by replacing S and ds with G/H and a G-invariant measure on G/H respectively (cf. [12]). We can treat this case in our scheme, because the integral over G/H can be regarded as the one over $S = \sigma_0(G/H)$ where $\sigma_0 : G/H \rightarrow G$ is a flat section of the fiber bundle $G \rightarrow G/H$.

These considerations are based on the existence of the discrete series of G or G/H, so it seems to be difficult to unfold the same process in the case that G has no such representations. One approach to treat the case is to find a non flat Borel section $\sigma : G/H \rightarrow G$. In the case of the Poincaré group and the affine Weyl-Heisenberg group, Ali, Antoine, and Gazeau [1] and Kalisa and Torrésani [10] respectively find a non square-integrable reprepentation $(\pi, \mathscr{H})$, a ψ in $\mathscr{H}$, and a non flat section σ such that (2) holds for π, ψ, and $S = \sigma(G/H)$. In this paper we shall investigate a transform associated to a principal series representation of noncompact semisimple Lie groups and we obtain a generalization of the Grossmann-Morlet transform and the Carderón identity. A transform associated to the analytic continuation of the holomorphic discrete series and its limit will be treated in the forthcoming paper [9].

2. Principal series representations. Let G be a noncompact connected semisimple Lie group with finite center and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a}_0 + \mathfrak{n}_0$ an Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of G. According to the process in [4, §6], we shall define a standard parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$. Let Σ be the set of roots of $(\mathfrak{g}, \mathfrak{a}_0)$ positive for $\mathfrak{n}_0$ and Σ_0 the subset of Σ consisting of simple roots. For each $F \subset \Sigma_0$ we set $\mathfrak{a} = \mathfrak{a}_F = \{H \in \mathfrak{a}_0 ; \alpha(H) = 0 \text{ for all } \alpha \in F\}$ and $\mathfrak{n} = \mathfrak{n}_F = \sum_{\alpha\in\Sigma\setminus\Sigma_F} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is the root space corresponding to α. Then the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is given by $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ where $Z_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{m}$

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