Integrability of Infinitesimal Automorphisms of Linear Poisson Manifolds

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1. Introduction. In the present paper, we discuss the integrability of infinitesimal automorphisms of linear Poisson manifolds. An infinitesimal automorphism X is said to be *integrable*, if it is a *Hamiltonian vector field.*

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. The linear Poisson structure on $\mathfrak{g}^*$ is defined as a Lie algebra structure on $C^\infty(\mathfrak{g}^*)$ satisfying Leibniz identity. This is equal to giving an antisymmetric contravariant 2-tensor P on $\mathfrak{g}^*$ which satisfies Jacobi identity. More precisely, for all $f, g \in C^\infty(\mathfrak{g}^*)$ and $\mu \in \mathfrak{g}^*$, the Poisson bracket is given by

$$\{f, g\}(\mu) = \langle \mu, [d_\mu f, d_\mu g] \rangle,$$

where $[\ ,\ ]$ is the Lie algebra operation in $\mathfrak{g}$, $\langle , \rangle$ is the pairing of $\mathfrak{g}^*$ with $\mathfrak{g}$, and $d_\mu f$ is the differential of f considered as an element of $\mathfrak{g}$ instead of $\mathfrak{g}^{**}$. In the case of general Poisson manifolds, the Poisson bracket is given by $\{f, g\} = \langle P \mid df \wedge dg \rangle$.

We denote by $G \cdot \mu$ the G-orbit passing through $\mu \in \mathfrak{g}^*$ with respect to the coadjoint representation of G on $\mathfrak{g}^*$. By the theorem of Kirillov-Kostant-Souriau, each $G \cdot \mu$ is a symplectic leaf in $\mathfrak{g}^*$. (Hence it is even dimensional.) Let G_μ be the isotropy group at μ. Then $G \cdot \mu$ is diffeomorphic to G/G_μ. For more informations about linear Poisson manifolds, see [7].

Now we shall define three (infinite dimensional) Lie algebras of vector fields on $\mathfrak{g}^*$. By an infinitesimal automorphism of $\mathfrak{g}^*$, we mean a smooth vector field X on $\mathfrak{g}^*$ such that $L(X)P = 0$, where $L(X)$ denotes the Lie derivative along X. We denote by $\mathscr{L}$ the Lie algebra consisting of such vector fields X. Let $\mathscr{I}$ be a Lie subalgebra of $\mathscr{L}$ consisting of vector fields X such that each X is tangent to symplectic leaves $G \cdot \mu$. Given $f \in C^\infty(\mathfrak{g}^*)$, $\{f, \cdot\}$ defines a derivation of $C^\infty(\mathfrak{g}^*)$. Hence there corresponds a vector field ξ_f, which we call the *Hamiltonian vector field.* And we denote by $\mathscr{H}$ the Lie algebra of Hamiltonian vector fields. Then there are canonical inclusions: $\mathscr{L} \supset \mathscr{I} \supset \mathscr{H}$. Direct calculation shows that both Lie subalgebras $\mathscr{I}$ and $\mathscr{H}$ are ideals of $\mathscr{L}$.

A vector field X of $\mathscr{L}$ is called "integrable" if it belongs to $\mathscr{H}$. If all vector fields of $\mathscr{L}$ are integrable (i.e. $\mathscr{L} = \mathscr{H}$), then $\mathscr{L}$ is called integrable. In the case of $\mathfrak{g} = \mathfrak{so}(3, R)$, we proved that $\mathscr{L}$ is integrable ([3] and [4]). In this paper, we treat the case of $\mathfrak{g} = \mathfrak{sl}(2, R)$.

Recall that the quotient space $\mathscr{L}/\mathscr{H}$ is nothing but the first Poisson cohomology ([1] and [5]). There are many papers about Poisson cohomology of "regular" Poisson manifolds ([1], [5], [6] and [8]). Note that linear Poisson manifolds give typical examples of "nonregular" Poisson manifolds. Therefore our study can be regarded as the first approach to the study of Poisson cohomology of "nonregular" Poisson manifolds.

2. Chevalley-Eilenberg complex. In this section, we shall express the integrability of vector fields in terms of Lie algebra cohomology (see for example [4]). Let (V, ρ) be any representation of the Lie algebra $\mathfrak{g}$ on a vector space. Associated to this representation, there is the Chevalley-Eilenberg complex:

$$V \xrightarrow{\partial_0} V \otimes \Lambda^1 \mathfrak{g}^* \xrightarrow{\partial_1} V \otimes \Lambda^2 \mathfrak{g}^*$$

where coboundary operators are defined by setting

$$(\partial_0 \alpha)(\xi_1) = \rho(\xi_1)(\alpha),$$
$$(\partial_1 \beta)(\xi_1 \wedge \xi_2) = \rho(\xi_1)(\beta(\xi_2)) - \rho(\xi_2)(\beta(\xi_1)) - \beta([\xi_1, \xi_2]),$$

for all $\alpha \in V$ and $\beta \in V \otimes \Lambda^1 \mathfrak{g}^*$ and $\xi_1, \xi_2 \in \mathfrak{g}$. It holds that $\partial_1 \cdot \partial_0 = 0$. The quotient $H^1(\mathfrak{g}; (V, \rho)) = \text{kernel } (\partial_1)/\text{image } (\partial_0)$ is called the first cohomology group of $\mathfrak{g}$ with coefficients in the module (V, ρ). Recall that when $\mathfrak{g}$ is semisimple and V is finite dimensional, the space $H^1(\mathfrak{g}; (V, \rho))$ vanishes. Let $x_1, x_2, \ldots, x_n$ be the basis of $\mathfrak{g}$. Then $x_1, x_2, \ldots, x_n$ are considered as coordinate functions on $\mathfrak{g}^*$. We denote by $F\{\mathfrak{g}\}$ the space of all formal functions with variables $x_1, x_2, \ldots, x_n$. $F\{\mathfrak{g}\}$ can be identified with the set