Warped Products with Critical Riemannian Metric*)

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1. Introduction. Let (B, g) and $(F, \bar{g})$ be two Riemannian manifolds of dimensions n and p respectively, and let f be a positive smooth function on B. Then the warped product space $M = B \times {}_f F$ is defined by the Riemannian metric $\tilde{g} = \pi^*(g) + (f \circ \pi)^2 \sigma^*(\bar{g})$, where π and σ are the projections of $B \times F$ onto B and F, respectively.

Let $n + p = m$. For a local coordinate system (x^a) $(a = 1, 2, \cdots, n)$ of B, the metric tensor g has the components (g_{ab}) and $\bar{g}$ on F has the components $(\bar{g}_{\alpha\beta})$ for a local coordinate system (y^α) $(\alpha = 1, 2, \cdots, p)$. Hence the metric tensor $\tilde{g}$ on M has the components

$$(\tilde{g}_{ji}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & f^2 \bar{g}_{\alpha\beta} \end{pmatrix}$$

with respect to the local coordinate system $x^i = (x^a, y^\alpha)$ on M and $i, j = 1, \cdots, m$.

Let ∇_b (resp. ∇_α) be the components of the covariant derivative with respect to g (resp. $\bar{g}$) and $\left\{ {a \atop b\ c} \right\}$ $\left(\text{resp. } \overline{\left\{ {\alpha \atop \beta\ \gamma} \right\}}\right)$ the christoffel symbol of B (resp. F). Then the christoffel symbol $\widetilde{\left\{ {i \atop j\ k} \right\}}$ of M are given as follows

(1.1) $\widetilde{\left\{ {c \atop b\ a} \right\}} = \left\{ {c \atop b\ a} \right\}$,

(1.2) $\widetilde{\left\{ {\alpha \atop d\ \gamma} \right\}} = \dfrac{(\nabla_d f)}{f} \delta_\gamma{}^\alpha$,

(1.3) $\widetilde{\left\{ {a \atop \delta\ \beta} \right\}} = -f(\nabla_b f) g^{ab} \bar{g}_{\delta\beta}$,

(1.4) $\widetilde{\left\{ {\gamma \atop \beta\ \alpha} \right\}} = \overline{\left\{ {\gamma \atop \beta\ \alpha} \right\}}$,

and the others are zero.

Let $\tilde{R}$, R, and $\bar{R}$ be the curvature tensor of M, B and F respectively, then we get [2, 3, 4, 5]

(1.5) $\tilde{R}_{dcb}{}^a = R_{dcb}{}^a$

(1.6) $\tilde{R}_{d\gamma b}{}^\alpha = \dfrac{1}{f}(\nabla_d f_b) \delta_\gamma{}^\alpha$

(1.7) $\tilde{R}_{\delta\gamma\beta}{}^\alpha = \bar{R}_{\delta\gamma\beta}{}^\alpha - \| f_e \|^2 (\delta_\delta{}^\alpha \bar{g}_{\gamma\beta} - \delta_\gamma{}^\alpha \bar{g}_{\delta\beta})$

and the others are zero, where $f_b = \nabla_b f$.

The components of Ricci tensors are given by

(1.8) $\tilde{S}_{cb} = S_{cb} - \dfrac{p}{f}(\nabla_c f_b)$,

(1.9) $\tilde{S}_{c\beta} = 0$,

(1.10) $\tilde{S}_{\gamma\beta} = \bar{S}_{\gamma\beta} - (p-1) \| f_e \|^2 \bar{g}_{\gamma\beta} - f \Delta f \bar{g}_{\gamma\beta}$,

where Δf is the Laplacian of f for g and $\tilde{S}$, S and $\bar{S}$ are the Ricci tensors of M, B and F respectively.

Let $\tilde{\gamma}$, γ and $\bar{\gamma}$ be the scalar curvatures of M, B and F respectively, then we have

(1.11) $\tilde{\gamma} = \gamma + f^{-2}\bar{\gamma} - 2pf^{-1}\Delta f - p(p-1) f^{-2} \| f_e \|^2$.

2. Critical Riemannian metrics. Let $(M = B \times {}_f F, \tilde{g})$ be a compact oriented Riemannian manifold. Consider the following Riemannian functional

$$H(\tilde{g}) = \int_M \tilde{\gamma}^2 \, d\mu,$$

where $d\mu$ is the volume element measured by $\tilde{g}$. A critical point of $H(\tilde{g})$ is called a critical Riemannian metric on M. In particular, every Einstein metric is a critical metric for H on M.

M. Berger [1] obtained the equation of the critical Riemannian metric in the following form in the tensor notations

(2.1) $$H_{ji} = c\, \tilde{g}_{ji},$$

where c is undetermined constant and H_{ji} is given by

(2.2) $H_{ji} = 2 \tilde{\nabla}_j \tilde{\nabla}_i \tilde{\gamma} - (\tilde{\Delta}\tilde{\gamma}) \tilde{g}_{ji} - 2\tilde{\gamma} \tilde{S}_{ji} + \dfrac{1}{2} \tilde{\gamma}^2 \tilde{g}_{ji}$,

where $\tilde{\nabla}$ means covariant differentiation with respect to $\tilde{g}$ and $\tilde{\Delta}\tilde{\gamma}$ is the Laplacian of $\tilde{\gamma}$ for $\tilde{g}$.

If the Riemannian metric $\tilde{g}$ on M is a critical Riemannian metric, then the undetermined constant c is determined as

(2.3) $$c = 2\left(\frac{1}{m} - 1\right)\tilde{\Delta}\tilde{\gamma} + \left(\frac{1}{2} - \frac{2}{m}\right)\tilde{\gamma}^2.$$

Hence, by use of (2.2) and (2.3), we have

Lemma 2.1. The Riemannian metric $\tilde{g}$ on warped product space $M = B \times {}_f F$ is critical Riemannian metric if and only if

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