The Diophantine Equation $a^x + b^y = c^z$. II

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§1. Introduction. In the previous paper [8], we proposed the following:

Conjecture. *If a, b, c, p, q, r are fixed positive integers satisfying $a^p + b^q = c^r$ with $p, q, r \geq 2$ and $(a, b) = 1$, then the Diophantine equation*

(1) $$a^x + b^y = c^z$$

has the only positive integral solution $(x, y, z) = (p, q, r)$.

When $(p, q, r) = (2,2,2)$, the above Conjecture is called Jeśmanowicz's conjecture. It has been verified that this conjecture holds for many Pythagorean numbers (cf. Jeśmanowicz [3], Takakuwa and Asaeda [5], [6], Takakuwa [7], Adachi [1]).

In [8], we considered the above Conjecture when $(p, q, r) = (2,2,3)$ and showed that it holds for certain a, b, c satisfying $a^2 + b^2 = c^3$.

In this paper, we consider the case $(p, q, r) = (2,2,5)$. Using an argument similar to the one used in [8], we shall prove that the above Conjecture also holds for certain a, b, c satisfying $a^2 + b^2 = c^5$ as specified in Theorem in §2. We shall also give some examples of a, b, c satisfying the conditions of Theorem.

§2. Theorem. We first prepare some lemmas.

In the same way as in the proof of Lemma 1 in [8], we obtain the following:

Lemma 1. *The integral solutions of the equation $a^2 + b^2 = c^5$ with $(a, b) = 1$ are given by*

$$a = \pm u(u^4 - 10u^2v^2 + 5v^4),$$
$$b = \pm v(5u^4 - 10u^2v^2 + v^4),\ c = u^2 + v^2,$$

where u, v are integers such that $(u, v) = 1$ and $u \not\equiv v \pmod 2$.

In the following, we consider the case $u = m$, $v = 1$; i.e.

(2) $$a = m(m^4 - 10m^2 + 5),\quad b = 5m^4 - 10m^2 + 1,\ c = m^2 + 1$$

and

$$m \text{ is even.}$$

Lemma 2. *Let a, b, c be positive integers satisfying* (2). *If the Diophantine equation* (1) *has positive integral solutions (x, y, z), then x and y are even.*

Proof. It suffices to show that

$$\left(\frac{a}{b}\right) = -1,\ \left(\frac{c}{b}\right) = 1,\ \left(\frac{b}{a'}\right) = -1 \text{ and } \left(\frac{c}{a'}\right) = 1$$

with $a = ma'$, where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. These imply that x and y are even.

Since $b \equiv 1 \pmod 8$, we have $\left(\frac{m}{b}\right) = 1$. In fact, putting $m = 2^s t$ ($s \geq 1$ and t is odd), $\left(\frac{m}{b}\right) = \left(\frac{2^s}{b}\right)\left(\frac{t}{b}\right) = \left(\frac{t}{b}\right) = \left(\frac{b}{t}\right) = \left(\frac{1}{t}\right) = 1$.

Hence we have $\left(\frac{a}{b}\right) = \left(\frac{m}{b}\right)\left(\frac{a'}{b}\right) = \left(\frac{a'}{b}\right) = \left(\frac{b}{a'}\right) = \left(\frac{5m^4 - 10m^2 + 1}{m^4 - 10m^2 + 5}\right) = \left(\frac{2}{m^4 - 10m^2 + 5}\right)$ $\left(\frac{5m^2 - 3}{m^4 - 10m^2 + 5}\right) = (-1) \cdot \left(\frac{m^4 - 10m^2 + 5}{5m^2 - 3}\right) = (-1) \cdot 1 = -1$. Thus we obtain $\left(\frac{a}{b}\right) = \left(\frac{b}{a'}\right) = -1$.

We also have $\left(\frac{c}{b}\right) = \left(\frac{b}{c}\right) = \left(\frac{16}{m^2 + 1}\right) = 1$, and $\left(\frac{c}{a'}\right) = \left(\frac{a'}{c}\right) = \left(\frac{16}{m^2 + 1}\right) = 1$. Q.E.D.

Lemma 3. *Let a, b, c be positive integers satisfying $a^2 + b^2 = c^5$ and $(a, b) = 1$. Suppose that there is an odd prime l such that $ab \equiv 0 \pmod l$ and $e \equiv 0 \pmod 5$, where e is the order of c modulo l. If the Diophantine equation* (1) *has positive integral solutions (x, y, z), then $z \equiv 0 \pmod 5$.*

Proof. We may suppose that $b \equiv 0 \pmod l$ without loss of generality.

It follows from $a^2 + b^2 = c^5$ that $a^2 \equiv c^5 \pmod l$. By (1), we see that $a^x \equiv c^z \pmod l$, so $c^{2z} \equiv a^{2x} \equiv c^{5x} \pmod l$. Hence we have $c^{5x-2z} \equiv 1 \pmod l$, which implies $5x - 2z \equiv 0 \pmod e$. Therefore we have $z \equiv 0 \pmod 5$. Q.E.D.

Lemma 4. (a) (Lebesgue [4]). *The Diophan-*