21. *A Skorokhod Problem with Singular Drift and its Application to the Origin of Universes*

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(Communicated by Kiyosi ITÔ, M. J. A., April 12, 1994)

1. Introduction. Let $R(t)$ be strictly increasing and continuous in $t \geq 0$ with $R(0) = 0$. In a space-time domain

(1.1) $$D = \{(t, x) ; t > 0, x \in [-R(t), R(t)]\},$$

we consider a singular diffusion equation and its formal adjoint

(1.2) $$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial x^2} + \frac{x}{t}\frac{\partial u}{\partial x} = 0, \quad -\frac{\partial \mu}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \mu}{\partial x^2} - \frac{\partial}{\partial x}\left(\frac{x}{t}\mu\right) = 0,$$

with the reflecting boundary condition. (1.2) determines a transition probability $Q(s, x ; t, dy)$, $s, t \in [a, b]$, $0 < a < b < \infty$. Since $\{Q(s, x ; t, dy) ; s \in (0, \varepsilon]\}$ is tight because of (1.1), we can chose $\xi(s) \downarrow 0$ so that

(1.3) $$Q^{\xi}(0, 0 ; t, dy) = \lim_{s \downarrow 0} Q(s, \xi(s) ; t, dy)$$

exists, but the limit $Q^{\xi}(0, 0 ; t, dy)$ depends on ξ and is not uniquely determined in general. We will discuss this problem and its implication to the origin of universes in terms of a Skorokhod problem with singular drift x/t.

2. A Skorokhod problem. Instead of (1.2) with the moving reflecting boundary we consider a two-sided Skorokhod problem with singular drift

(2.1) $$X_t = \sigma\beta_t + \int_0^t \frac{X_s}{s} ds + \Phi_t, \quad |X_t| \leq R(t),$$

where β_t denotes a one-dimensional Brownian motion, and

(2.2) $$\begin{gathered} \Phi_t \text{ is continuous in } t \geq 0,\ \Phi_0 = 0, \\ \Phi_t = \Phi_t^{(-)} - \Phi_t^{(+)}, \text{ for } t > 0, \\ \Phi_t^{(-)} \text{ increases only on } \{s : X(s) = -R(s)\}, \\ \Phi_t^{(+)} \text{ increases only on } \{s : X(s) = R(s)\}. \end{gathered}$$

We will construct solutions of the problem (2.1), and show that the shape of the boundary of the domain D influences the uniqueness and non-uniqueness of solutions of (2.1). Assuming

(2.3) $$R(t) = (\alpha t)^{\gamma},\ 0 < \gamma < 1, \text{ for small } t,$$

where $\alpha > 0$, we shall analyze the behaviour of solutions near the origin.

3. The case without boundary. Equation (1.2) but $[a, b] \times \mathbf{R}$ without boundary determines another transition probability $\mathrm{P}(s, x ; t, dy)$. Contrary to the case with reflecting boundary, $\mathrm{P}(0, 0 ; t, dy)$ cannot be well-defined, since $\{\mathrm{P}(s, x ; t, dy) : s \in (0, \varepsilon]\}$ is not tight. Hence, a stochastic differential equation

(3.1) $$X_t = \sigma\beta_t + \int_0^t \frac{X_s}{s} ds$$

has no adapted solution, where β_t denotes a one-dimensional Brownian motion. Nevertheless, a theorem of Jeulin-Yor [5] (cf. [6]) claims that X_t satisfies