15. *Commuting Families of Symmetric Differential Operators*

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Introduction. Many commuting families of differential operators or completely integrable quantum systems have been constructed in connection with root systems (cf. [10] and references therein). Such families often have a certain symmetry in coordinates.

The radial parts of invariant differential operators on symmetric spaces give a good example of a commuting family of differential operators (cf. [1]). In this case some parameters take only some discrete values determined by the dimensions of the root spaces for the symmetric spaces.

On the other hand, [12] generalized the example to have holomorphic parameters if the root system is of type A_n. The same generalization was given by [2], [3], [4], [7], [8] in general root systems. If the root system is of classical type, their operators give examples of the commuting families studied in this note (cf. Remark 3 iii)). Namely we shall determine all the families under the assumption of a symmetry in coordinates.

Let W be the Weyl group of type A_{n-1} with $n \geq 3$ or of type B_n with $n \geq 2$ or of type D_n with $n \geq 4$. We identify W with the group of the coordinate transformations

$$(x_1,\ldots,x_n) \mapsto (\varepsilon_1 x_{\sigma(1)},\ldots,\varepsilon_n x_{\sigma(n)})$$

of $\boldsymbol{R}^n$, where σ are the elements of the n-th symmetric group $\mathfrak{S}_n$ and

$$\begin{cases} \varepsilon_1 = \cdots = \varepsilon_n = 1 & \text{if } W \text{ is of type } A_{n-1}, \\ \varepsilon_1 = \pm 1,\cdots,\varepsilon_n = \pm 1 & \text{if } W \text{ is of type } B_n, \\ \varepsilon_1 = \pm 1,\cdots,\varepsilon_n = \pm 1 \text{ and } \#\{i\,;\,\varepsilon_i = -1\} \text{ is even} & \text{if } W \text{ is of type } D_n. \end{cases}$$

We examine the Laplacian

$$P = -\frac{1}{2}\sum_{1\leq j\leq n}\frac{\partial^2}{\partial x_j^2} + V(x)$$

on $\boldsymbol{R}^n$ with a W-invariant potential $V(x)$ which has enough W-invariant commuting differential operators. To be precise we assume that there exist W-invariant differential operators $P_1,\ldots,P_n$ with

$$[P_i, P_j] = 0 \quad \text{for} \quad 1 \leq i < j \leq n$$

such that

$$\begin{cases} P = P_2 - \dfrac{1}{2}P_1^2, \\ P_j = \displaystyle\sum_{1\leq i_1<\cdots<i_j\leq n} \partial_{i_1}\cdots\partial_{i_j} + R_j \quad \text{with ord } R_j < j \quad \text{for} \quad 1 \leq j \leq n \end{cases}$$

or

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