13. *Associated Varieties and Gelfand-Kirillov Dimensions for the Discrete Series of a Semisimple Lie Group*

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(Communicated by Kiyosi ITÔ, M. J. A., Feb. 14, 1994)

1. Introduction. Let G be a connected semisimple Lie group with finite center, and K be a maximal compact subgroup of G. The corresponding complexified Lie algebras are denoted respectively by $\mathfrak{g}$ and $\mathfrak{k}$. We assume Harish-Chandra's rank condition $\operatorname{rank} G = \operatorname{rank} K$, which is necessary and sufficient for G to have a non-empty set of discrete series, or of square-integrable irreducible unitary representations of G.

In this paper, we describe the associated varieties of Harish-Chandra $(\mathfrak{g}, K)$-modules of discrete series, by an elementary and direct method based on [3]. The description is as in

Theorem 1. *If H_Λ is the $(\mathfrak{g}, K)$-module of discrete series with Harish-Chanda parameter $\Lambda = \lambda + \rho_c - \rho_n$ (see §3), then its associated variety $\mathcal{V}(H_\Lambda) \subset \mathfrak{g}$ (see §2) coincides with the nilpotent cone $K_C\mathfrak{p}_-$, which is equal to $\operatorname{Ad}(K)\mathfrak{p}_-$. Here K_C denotes the analytic subgroup of adjoint group $G_C := \operatorname{Int}(\mathfrak{g})$ of $\mathfrak{g}$, with Lie algebra $\mathfrak{k}$, and $\mathfrak{p}_- = \sum_{\beta\in\Delta_n^-}\mathfrak{g}_\beta$ is the sum of root subspaces $\mathfrak{g}_\beta$ of $\mathfrak{g}$ corresponding to the noncompact roots β such that $(\Lambda, \beta) < 0$.*

We further give in Theorem 4 an explicit formula for the Gelfand-Kirillov dimensions $d(H_\Lambda) \dim \mathcal{V}(H_\Lambda)$ of discrete series in the case of unitary groups $G = SU(p, q)$, by specifying the unique nilpotent G_C-orbits in $\mathfrak{g}$ which intersect $\mathfrak{p}_-$ densely. Note that this important invariant $d(H_\Lambda)$ coincides with the degree of Hilbert polynomial of H_Λ.

We know that Theorem 1 can be deduced from deep results in [1, III] and [4] by passing to D-module via Beilinson-Bernstein correspondence. However, the associated variety is an object attached directly to each finitely generated $U(\mathfrak{g})$-module. From this reason, we give here a direct path to the theorem avoiding the above detour by D-module. Our proof of Theorem 1 is simple in the sense that it uses only some basic results of [3] on the realization of H_Λ as the kernel space of differential operator $\mathcal{D}_\lambda$ on G/K of gradient-type. Nevertheless, this method gives us new conclusions also (Theorem 3). For instance, we find that the associated variety of discrete series can be expressed in terms of the symbol mapping of $\mathcal{D}_\lambda$.

2. Associated varieties for $U(\mathfrak{g})$-modules. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and $(U_k(\mathfrak{g}))_{k=0,1,..}$ be the natural increasing filtration of $U(\mathfrak{g})$, with $U_k(\mathfrak{g})$ the subspace of $U(\mathfrak{g})$ generated by elements $X^m (0 \leq m \leq k$, $X \in \mathfrak{g})$. We identify the associated graded ring $\operatorname{gr} U(\mathfrak{g}) = \bigoplus_{k\geq 0} U_k(\mathfrak{g}) / U_{k-1}(\mathfrak{g})$ $(U_{-1}(\mathfrak{g}) := (0))$ with the symmetric algebra $S(\mathfrak{g}) = \bigoplus_{k\geq 0} S^k(\mathfrak{g})$ of $\mathfrak{g}$ in the canonical way. Here $S^k(\mathfrak{g})$ denotes the homogeneous component of