55. *Complements to the Furuta Inequality*[†)]

By Masatoshi FUJII[*)], Takayuki FURUTA[**)], and Eizaburo KAMEI[***)]

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Abstract: Complementary results to the Furuta inequality are given in cases of positive invertible operators.

§1. Introduction. In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (in symbol: $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. Also an operator T is strictly positive (in symbol: $T > 0$) if T is positive and invertible.

As an extension of the Löwner-Heinz theorem [12][10], we established the following Furuta inequality [4].

Theorem A (Furuta inequality). *If $A \geq B \geq 0$, then for each $r \geq 0$,*

(i) $$(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q}$$

and

(ii) $$(A^r A^p A^r)^{1/q} \geq (A^r B^p A^r)^{1/q}$$

hold for p and q such that $p \geq 0$ and $q \geq 1$ with $(1+2r)q \geq p+2r$.

Alternative proofs of Theorem A are given in [1][5] and [11] and also one page proof is shown in [6]. Recently it turns out that Theorem A has a lot of applications, in fact [2][3][7][8] and [9] are some of them.

We remark that the Furuta inequality yields the following famous Löwner-Heinz inequality when we put $r=0$ in (i) or (ii) of Theorem A;

Theorem B (Löwner-Heinz inequality).

(*) $$A \geq B \geq 0 \text{ ensures } A^\alpha \geq B^\alpha \text{ for any } \alpha \in [0,1].$$

§2. Statement of results. Theorem 1. *If $A \geq B > 0$, then*

$$(B^r A^\alpha B^r)^\beta \geq (B^r B^\alpha B^r)^\beta$$

holds under any one of the following conditions;

(i) $$\frac{1}{\beta} \leq \alpha,\ 0 < \beta < 1, \text{ and } \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$$

(ii) $$\frac{1}{\beta} \leq \alpha \leq 1,\ 1 < \beta \leq 2, \text{ and } \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}$$

(iii) $$\frac{1}{2} \leq \alpha \leq 1,\ 2 \leq \beta, \text{ and } \gamma = \frac{\alpha\beta - 1}{2(1-\beta)}.$$

Remark 1. (i) and (ii) are announced in [13, p. 61], but in the proof of Theorem 1 under below we remark that (i) is nothing but exchange of parameters p, q and r in Theorem A and a simple proof of (ii) can be obtained along a method of [6] by using polar decomposition. In this paper we shall

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[*)] Department of Mathematics, Osaka Kyoiku University.
[**)] Department of Applied Mathematics, Science University of Tokyo.
[***)] Momodani Senior High School, Osaka.