# 51. *Triangles and Elliptic Curves. II*

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This is a continuation of my preceding paper [1] which will be referred to as (I) in this paper. In (I), to each parameter $t = (a, b, c)$, we associated a pair $(E_t, \pi_t)$ of an elliptic plane curve and a point on it. In this paper, we shall find an elliptic space curve $C$ in a fibre of the map $t \mapsto E_t$ so that the map $t \mapsto \pi_t$ is an isogeny: $C \to E = E_t$, $t \in C$. As in (I), this paper will contain an assertion on the Mordell-Weil group $E(k)$ when $k$ is a number field.

**§1. Space *T*.** Let $k$ be a field of characteristic $\neq 2$ and $\bar{k}$ be the algebraic closure of $k$. Let $l = l(t)$, $m = m(t)$, $n = n(t)$ be independent linear forms on the vector space $\bar{k}^3$. Our parameter space is defined by

(1.1) $$T = \{t \in \bar{k}^3 \,;\, (l^2 - m^2)(m^2 - n^2)(n^2 - l^2) \neq 0\}.$$

For each $t \in T$, put

(1.2) $$P_t = (l^2 - n^2) + (m^2 - n^2),$$

(1.3) $$Q_t = (l^2 - n^2)(m^2 - n^2).$$

Then we have

(1.4) $$P_t^2 - 4Q_t = (l^2 - m^2)^2.$$

By the definition of $T$, we obtain elliptic curves

(1.5) $$E_t : y^2 = x^3 + P_t x^2 + Q_t x = x(x - (n^2 - l^2))(x - (n^2 - m^2)), \quad t \in T.$$

One verifies easily that

(1.6) $$\pi_t = (n^2, lmn) \in E_t, \quad t \in T.$$

If forms $l$, $m$, $n$ have coefficients in $k$ and if $t \in T(k) = T \cap k^3$, then the elliptic curve $E_t$ is defined over $k$ and $\pi_t \in E_t(k) = E_t \cap k^2$.

(1.7) **Example.** If we put $l(t) = (b + a)/2$, $m(t) = (b - a)/2$, $n(t) = c/2$, for $t = (a, b, c) \in T$, then we find ourselves in the situation of (I): $P_t = (a^2 + b^2 - c^2)/2$, $Q_t = (a + b + c)(a + b - c)(a - b + c)(a - b - c)/16$ and $\pi_t = (c^2/4, c(b^2 - a^2)/8)$.

(1.8) **Example.** In §2 we shall meet the simplest situation where $l(t) = a$, $m(t) = b$, $n(t) = c$. In this case, we have $P_t = a^2 + b^2 - 2c^2$, $Q_t = (a^2 - c^2)(b^2 - c^2)$ and $\pi_t = (c^2, abc)$.

Back to general $l$, $m$, $n$, we shall consider the equivalence relation in $T$ defined by

(1.9) $$t \sim t' \Leftrightarrow E_t = E_{t'}, \quad t, t' \in T.$$

In other words,

(1.10) $$t \sim t' \Leftrightarrow P_t = P_{t'}, \quad Q_t = Q_{t'}, \quad t, t' \in T.$$

Now call $t_0$ a point in $T$ fixed once for all and consider the class $F$ containing $t_0$:

(1.11) $$F = \{t \in T \,;\, t \sim t_0\}.$$