

1. On Real Submanifolds of Kähler Manifolds Foliated by Complex Submanifolds

By Takashi INABA^{*)} and Michael A. MISHCHENKO^{**)}

(Communicated by Heisuke HIRONAKA, M. J. A., Jan. 12, 1994)

This work is motivated by the study of topological properties of Levi-flat hypersurfaces of complex manifolds (see e.g. [1] and [2]). Let X be a Kähler manifold and M a compact orientable real submanifold of X . We suppose that M admits a foliation \mathcal{F} whose leaves are complex submanifolds of X . In this setting, we are interested in investigating the relation between the topology of \mathcal{F} and the complex structure of X . We first observe that the exactness of the Kähler form implies the intricacies of \mathcal{F} (Theorem 1 and Remark), and then apply it to the nonexistence problem of Levi-flat hypersurfaces in CP^2 (Corollary 2).

Theorem 1. *Let X , M and \mathcal{F} be as above, and let $p = \dim_{\mathbb{C}} \mathcal{F}$. If the p -th power of the Kähler form of X is exact when restricted to M , then \mathcal{F} has no nontrivial foliation cycles.*

See e.g. [5] for the definition of foliation cycles.

Proof. Suppose \mathcal{F} has a nontrivial foliation cycle C . Since each leaf of \mathcal{F} is a complex submanifold of X , the p -th power of the Kähler form ω of X restricts to a volume form on each leaf. Then, by the local integration formula of foliation cycles ([5, Theorem I.12]) we see that the nontriviality of C implies the nonvanishing of the homological pairing $\langle C, \omega^p | M \rangle$. Thus, $\omega^p | M$ cannot be exact and the proof is complete.

Remark. It is known ([3], [5]) that if a foliation \mathcal{F} on a compact manifold M admits no nontrivial foliation cycles, \mathcal{F} must satisfy the following properties:

(1) Every leaf of \mathcal{F} has exponential growth. In particular, \mathcal{F} has no compact leaves.

(2) If $\text{codim } \mathcal{F} = 1$, some leaf of \mathcal{F} has nontrivial holonomy.

(3) If $\text{codim } \mathcal{F} = 1$ and $\dim M = 3$, $\pi_1(M)$ has exponential growth.

Corollary 1. *Let M be a compact smooth real hypersurface of CP^n ($n \geq 2$). Suppose that the Levi form on M has constant rank k ($0 \leq k \leq [n/2] - 1$) and hence defines a codimension $(2k + 1)$ foliation \mathcal{F} on M by complex submanifolds of CP^n ([4]). Then, \mathcal{F} has no nontrivial foliation cycles.*

Proof. We denote by $i : M \rightarrow CP^n$ the inclusion map and by ω the standard Kähler form of CP^n . By Theorem 1, it suffices to show that $i^* \omega^{n-k-1}$ is exact for $0 \leq k \leq [n/2] - 1$. Suppose the contrary. One can find a $2(n - k - 1)$ -cycle α of M such that $\langle i^* \omega^{n-k-1}, \alpha \rangle \neq 0$. Then, $\langle \omega^{n-k-1}, i_* \alpha \rangle \neq 0$. Hence, $i_*[\alpha]$ is nontrivial in $H_{2(n-k-1)}(CP^n; \mathbb{Z})$. From this it follows that the

^{*)} College of Arts and Sciences, Chiba University.

^{**)} Department of Mathematics, Moscow State University, Russia.